

What Community College Developmental Mathematics Students Understand about Mathematics

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The nation is facing a crisis in its community colleges: more and more students are attending community colleges, but most of them are not prepared for college-level work. The problem may be most dire in mathematics. By most accounts, the majority of students entering community colleges are placed (based on placement test performance) into "developmental" (or remedial) mathematics courses (e.g., Adelman, 1985; Bailey et al., 2005). The organization of developmental mathematics differs from school to school, but most colleges have a sequence of developmental mathematics courses that starts with basic arithmetic, then goes on to pre-algebra, elementary algebra, and finally intermediate algebra, all of which must be passed before a student can enroll in a transfer-level college mathematics course.

Because the way mathematics has traditionally been taught is sequential, the implications for students who are placed in the lower-level courses can be quite severe. A student placed in basic arithmetic may face two full years of mathematics classes before he or she can take a college-level course. This might not be so bad if they succeeded in the two-year endeavor. But the data show that most do not: students either get discouraged and drop out all together, or they get weeded out at each articulation point, failing to pass from one course to the next (Bailey, 2009). In this way, developmental mathematics becomes a primary barrier for students ever being able to complete a post-secondary degree, which has significant consequences for their future employment.

One thing not often emphasized in the literature is the role that our K-12 education system plays in this problem. We know from international studies that U.S.

mathematics education is mediocre at best when compared with other industrialized nations. But the fact that community college students, most of whom graduate from U.S. high schools, are not able to perform basic arithmetic, pre-algebra and algebra, shows the real cost of our failure to teach mathematics in a deep and meaningful way in our elementary, middle and high schools. Although our focus here is on the community college students, it is important to acknowledge that the methods used to teach mathematics in K-12 schools are not succeeding, and that the limitations of students' mathematical proficiency are cumulative and increasingly obvious over time.

The limitations in K-12 teaching methods have been well-documented in the research literature. The Trends In International Mathematics and Science Study (TIMSS) video studies (Stigler & Hiebert, 1999; Hiebert et al., 2003) showed that the most common teaching methods used in the U.S. focus almost entirely on practicing routine procedures with virtually no emphasis on understanding of core mathematics concepts that might help students forge connections among the numerous mathematical procedures that make up the mathematics curriculum in the U.S. The high-achieving countries in TIMSS, in contrast, use instructional methods that focus on actively engaging students with understanding mathematical concepts. Procedures are taught, of course, but are connected with the concepts on which they are based. In the U.S., procedures are more often presented as step-by-step actions that students must memorize, not as moves that make sense mathematically.

Given that U.S. students are taught mathematics as a large number of apparently unrelated procedures that must be memorized, it is not surprising that they forget most of them by the time they enter the community college. It is true that some

students figure out on their own that mathematics makes sense and that procedures forgotten can be reconstructed based on a relatively small number of core concepts. And even a few students who don't figure this out are smart enough to actually remember the procedures they are taught in school. But many students don't figure this out, and these are the ones that swell the ranks of students who fail the placement tests and end up in developmental mathematics.

Sadly, all the evidence we have (which is not much) shows that although community college faculty are far more knowledgeable about mathematics than are their K-12 counterparts (Lutzer et al., 2007), their teaching methods may not differ much from those used in K-12 schools (Grubb, 1999). "Drill-and-skill" is still thought to dominate most instruction (Goldrick-Rab, 2007). Thus, students who failed to learn how to divide fractions in elementary school, and who also probably did not benefit from attempts to re-teach the algorithm in middle and high school, are basically presented the same material in the same way yet again. It should be no surprise that the methods that failed to work the first time also don't work in community college. And yet that is the best we have been able to do thus far.

Currently there is great interest in reforming developmental mathematics education at the community college. Yet, it is worth noting that almost none of the reforms have focused on actually changing the teaching methods and routines that define the teaching and learning of mathematics in community colleges. Many schools have instituted courses that teach students how to study, how to organize their time, and how to have a more productive motivational stance towards academic pursuits (Zachry, 2008; Zeidenberg et al, 2007). They have tried to make it easier for students burdened with

families and full-time jobs to find time to devote to their studies. They have created forms of supplemental instruction (Blanc et al., 1983; Martin & Arendale, 1994) and learning assistance centers (Perin, 2004). They have tried to break down bureaucratic barriers that make it difficult for students to navigate the complex pathways through myriad courses that must be followed if students are ever to emerge from developmental math and pass a transfer-level course. Some have redesigned the curriculum - e.g., accelerated it, slowed it down, or tried to weed out unnecessary topics (e.g., Lucas & McCormick, 2007). Yet few have questioned the methods used to teach mathematics (Zachry, 2008).

An assumption we make in this report is that substantive improvements in mathematics learning will not occur unless we can succeed in transforming the way mathematics is taught. In particular, we are interested in exploring the hypothesis that these students who have failed to learn mathematics in a deep and lasting way up to this point might be able to do so if we can convince them, first, that mathematics makes sense, and then provide them with the tools and opportunities to think and reason. In other words, if we can teach mathematics as a coherent and tightly related system of ideas and procedures that are logically linked, might it not be possible to accelerate and deepen students' learning and create in them the disposition to reason about fundamental concepts? Might this approach reach those students who have not benefited from the way they have been taught mathematics up to this point (English & Halford, 1995)?

Consideration of this hypothesis led us to inquire into what we actually know about the mathematics knowledge and understanding of students who are placed into developmental math courses. Surprisingly, an extensive search of the literature revealed that we know almost nothing about these aspects of community college students. Grubb

(2005) made a similar point: we know quite a bit about community college teachers and about the institutions in which they work.

...but our knowledge of students and their attitudes toward learning is sorely lacking. ... The conventional descriptions of developmental students stress demographic characteristics (for example, first-generation college status and ethnicity) and external demands (such as employment and family), but aside from finding evidence of low self-esteem and external locus of control, there has been little effort to understand how developmental students think about their education. (Grubb & Cox, 2005, p. 95).

Most of what we know about the mathematical knowledge of community college students we learn from placement tests (Accuplacer, Compass, MDTP). But placement test data is almost impossible to come by due to the high-stakes nature of the tests and the need to keep items protected. Further, the most commonly used tests (Accuplacer and Compass) are adaptive tests, meaning that students take only the minimal items needed to determine their final score, and so don't take items that might give a fuller picture of their mathematical knowledge. Finally, most of the items on the placement tests are procedural in nature: they are designed to assess what students are able to do, but not what students understand about fundamental mathematical concepts.

Because of this gap in the literature, we undertook the small study reported here to gather information related to three questions:

- What do students actually understand about mathematics concepts that underlie the topics they've been taught?

- What do they think it means to DO mathematics? (remember vs. understand/reason)
- Can we get students to reason about mathematics (“If this is true, then that would have to be true...”) or are they stuck with just remembering procedures?

We investigated these three broad questions using several sources of data. The first data source comes from one of the placement tests to which we've referred, the Mathematics Diagnostic Testing Project (MDTP). The purpose of examining it was to see what we could glean about student understanding from an existing measure. The MDTP is unusual in that it is not a commercially designed or administered test, and it is not adaptive. It was developed by a group of mathematics professors back in the early 1980s, all of whom teach at public institutions of higher education in California. The goal of the test was not placement, initially, but was to give feedback to high schools on how well prepared their students were for entry in the University of California or California State University systems. But many community colleges do use the MDTP for placement purposes, including more than 50 in California. Interestingly, the items used on the MDTP tests have not changed since 1986. For this study we have been able to get access to all placement test data given by Santa Barbara City College for the past nearly 20 years. For the present report, we will present findings from the tests administered during the 2008-2009 academic year.

The second data source was a survey of math questions that we administered to a convenience sample of 748 community college developmental mathematics students. There were a total of twelve questions, and each student answered four. The purpose of

this survey was to delve more deeply into students' reasoning and to gather information that might help us in the design of the final data source, the one-on-one interviews.

The one-on-one interviews are now being conducted with community college developmental mathematics students. The goal of these interviews is to dig deeper in each case, trying to discern what, precisely, underlies each student's difficulties with mathematics. Is it simply failure to remember the conventions of mathematics? Is it a deficiency in basic knowledge of number and quantity? Is it a lack of conceptual understanding? What do these students understand about basic mathematics concepts (regardless of their ability to solve school-like mathematics problems)? Also, what do these students think it means to do mathematics? Is it just remembering, or is reasoning also required? And if we give them the chance, can they reason? Can they discover some new mathematical fact based only on making effective use of other facts they know?

More details on methods will be presented together with results in the following sections.

Placement Test Data

Participants and Tests

All Santa Barbara Community College students who took the Mathematics Diagnostic Testing Project (MDTP) placement tests during the 2008-2009 school year were included in the study. Tests were administered at three time points during the year: summer and fall of 2008, and spring of 2009. In all, 5830 tests were administered.

There were four different tests: Algebra Readiness, Elementary Algebra, Intermediate Algebra, and Pre-Calculus. Although the majority of students took only one test, some took more than one in order to determine their placement in the mathematics

sequence. As shown in Table 1, the gender of participants was relatively stable across tests, with slightly more males than females in each case. Ethnicity varied somewhat depending on test form, with the Hispanic and Black populations decreasing as test level increased. The Asian population increased as test level increased. Age decreased slightly with increase in test level.

Table 1. Sample size, age, gender, and ethnicity of Santa Barbara Community College students who took the MDTP placement tests during the 2008-2009 academic year.

Test	N	Age Mean (SD)	% male	Ethnicity					
				% White	% Hispanic	% Black	% Asian	% Other	% No response / missing
Algebra Readiness	1643	21 (6.3)	52	46	34	6	2	10	2
Elementary Algebra	1856	19 (3.5)	51	58	24	4	4	8	2
Intermediate Algebra	1651	19 (3.2)	56	59	21	2	7	8	3
Pre-Calculus	680	18 (2.2)	54	49	12	1	27	9	2

Items

There are 50 multiple choice items on the Algebra Readiness and Elementary Algebra assessments, 45 on the Intermediate Algebra assessment, and 60 on the Pre-Calculus assessment. The items on each assessment are grouped into multiple subscales, defined by the test writers. For the Algebra Readiness, Elementary Algebra, and Intermediate Algebra assessments, students had 45 minutes to complete the test. For the Pre-Calculus test students were allowed 60 minutes.

Student Difficulties

The examination of standardized test results often begins and ends with an analysis of mean scores. Our primary interest in the MDTP, however, lay not in the percent of items students got correct on the test or on a subscale of it, but rather in what their answer selections could tell us about their thinking. A correctly chosen response on a multiple choice test *may* indicate understanding. (That's an issue to be pursued with the interviews described below.) The selection of a wrong answer can sometimes be even more telling. Students occasionally answer questions randomly, but more often than not, they make their selection with some thought. Exploring the patterns of students' selections of wrong answers was therefore our starting point in identifying student difficulties.

Our examination of incorrect answers has focused thus far on the Algebra Readiness and Elementary Algebra assessments. For each we determined which items on the test proved most difficult for students. There were three criteria upon which our definition of difficulty was based. First, we included all items for which fewer than 25 percent of participants marked the correct answer. We also included items for which more students selected an incorrect answer than selected the correct answer. Finally, we counted those items for which there were two incorrect answer options selected by at least 20 percent of students. The result was a collection of 13 difficult items for Algebra Readiness and 10 difficult items for Elementary Algebra. Those items and the common errors made on them appear in Tables 2 and 3, respectively. It is important to note that the table describes common procedural errors. Errors in reasoning are described in a subsequent section.

Table 2. Difficult items on the Algebra Readiness form of the MDTP (in ascending order of percent correct), 2008-2009.

Item description	% of students who answered correctly	Common error(s)	% of students who made common error
Add a simple fraction and a decimal.	19	Converted decimal to a fraction, then added numerators and added denominators	28
Find LCM of two numbers.	21	Found GCF	59
Order four numbers (two simple fractions and two decimals).	22	Represented $1/3$ as $.3$ and ordered decimals by number of digits	24
		Converted fractions to decimals and ordered by number of digits	36
Add two squares under a radical.	23	Assumed $a^2 + b^2 = (a + b)^2$ or that $\sqrt{a^2 + b^2} = \sqrt{a^2} + \sqrt{b^2}$	25
		Added two squares, but failed to take the square root, stopping short of solving	31
Find a missing length for one of two similar triangles.	24	Multipled two bases and divided by the third	23
		Approximated ratio	25
Add two improper fractions.	24	Added numerators and added denominators	41
Find the missing value of a portion of a circle that has two portions labeled with simple fractions	26	Added numerators and denominators of the two fractions provided, stopping short of solving [other option was also stop short]	45
Find the diameter of a circle, given the area.	26	Found radius and failed to cancel π , stopping short of solving	37
Find the percent increase between two dollar amounts.	27	Found dollar amount increase and labeled it as a percentage, stopping short of solving	43
		Used larger of the two amounts as denominator when calculating increase	23
Find area of half of a square drawn on a	33	Found area of the square, stopping short of solving	28

coordinate plane.			
Find the largest of four simple fractions.	33	Found smallest fraction or converted to decimals and chose the only fraction that didn't repeat	44
Multiply two simple fractions.	37	Simplified incorrectly before multiplying	20
		Simplified incorrectly before multiplying	22
Divide one decimal by another.	41	Misplaced decimal (omitted zero as a placeholder)	23
		Divided denominator by numerator and misplaced decimal	20

Table 3. Difficult items on the Elementary Algebra form of the MDTP (in ascending order of percent correct), 2008-2009.

Item description	% of students who answered correctly	Common error(s)	% of students who made common error
Add two fractions that include variables	15	Added numerators and added denominators	34
Multiply two fractions that include variables.	16	Simplified incorrectly before multiplying	23
		Simplified incorrectly before multiplying and misplaced negative sign	24
Solve for x in a quadratic equation.	17	Factored the quadratic equation incorrectly and perhaps also solved for x incorrectly	21
Simplify a fraction that includes variables.	19	Simplified incorrectly	31
Find the percent that a larger number is of a smaller.	26	Divided the larger number by the smaller number, but failed to move the decimal in converting to a percent, stopping short of solving	24
		Divided the smaller number by the larger and converted the quotient to a decimal	25
Find the value of a number with a negative exponent.	26	Ignored the negative sign in the exponent	50

Find the area of a triangle inside a square given two lengths on the square.	31	Multiplied the two lengths provided	22
		Found the area of a triangle different from the one asked	23
Square a binomial expression.	32	Omitted the 'xy' term Squared each term in the expression and made an error with the negative sign	38
		Squared each term in the expression and omitted the 'xy' term	21
Find the difference between two square roots.	34	Factored the numbers provided and placed the common factor outside the radical without taking its square root	23
		Subtracted one number from the other and took square root of the difference	24
Identify the smallest of three consecutive integers given the sum of those integers	46	Constructed equation incorrectly	23

Although some difficulties were problem specific (e.g., confusing perimeter with area), a few core themes emerged when we examined the errors students made on the most difficult test items.

Several of the most common errors involved working with fractions. Across the two placement tests, the most common mistake was to simplify incorrectly. On the Algebra Readiness assessment, two of the frequent errors on difficult problems were caused by simplifying simple fractions incorrectly (e.g., simplifying $9/16$ as $3/4$).¹ On the Elementary Algebra assessment, three of the frequent errors on difficult problems were made when simplifying terms with variables (e.g., simplifying $(x + 1)/(x^2 + 5)$ as $1/(x +$

¹ In order to protect the items that appear on the MDTP, items are discussed in general terms and numbers have been changed.

4). In these cases the option chosen showed that either the students factored expressions incorrectly or made no attempt to use factoring.

It was also the case, as is common with younger students, that our community college sample frequently added across the numerator and across the denominator when adding fractions (e.g., $\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$). Three of the commonly chosen wrong answers we examined were caused by that mistake on the Algebra Readiness test and the process presented itself also on the Elementary Algebra assessment. Finally, the Algebra Readiness test also showed multiple instances of converting a fraction to a decimal by dividing the denominator by the numerator (e.g., $\frac{5}{8} = 8 \div 5$). These errors reveal that rather than using number sense, students rely on a memorized procedure, only to carry out the procedure incorrectly or inappropriately.

Answer choices related to decimals lead us to think that students may not have a firm grasp of place value. For instance, two frequently chosen answer options suggested that students believed that the size of a value written in decimal form was determined by the number of digits in it (e.g., $0.53 < 0.333$).

Another emergent theme suggested that students do not know what operations are allowable on equations with exponents and square roots. For example, some students added terms that shared a common exponent (e.g., $4^2 + 5^2 = 9^2$). Others treated the radical as a parenthetical statement, extracting a common factor from the terms within two radicals (e.g., $\sqrt{15} + \sqrt{45} = 15\sqrt{3}$).

Two final themes were related not as much to procedural misunderstanding as they were to problem solving. It was common, particularly on the Algebra Readiness assessment, for students to respond to a multi-step problem by completing only the first

step. It was as if they knew the steps to take, but when they saw an intermediate response as an answer option, they aborted the solution process. "Stopping short" could be used to explain five of the common errors on difficult Algebra Readiness items and one error on a difficult Elementary Algebra item. Another possible interpretation is that the student knew the first step, and then knew there was some next step, but couldn't remember it and chose the option matching what s/he knew was correct.

Lastly, it appeared as though students sometimes fell back on their knowledge of how math questions are typically posed. It was as if the item (or answer options) prompted their approach to it. For instance, when asked to find the least common multiple of two numbers that also had a greatest common factor other than one, they selected the answer that represented the greatest common factor. For example, if asked for the least common multiple of 6 and 9, students answered 3 (the greatest common factor) instead of 18 (the correct answer). Rarely do students practice finding least common multiples on anything but numbers *without* common factors, so they assumed in this case that the question was actually seeking the greatest common factor.

Students also fell back on what they're typically asked to do when they were presented with a percentage to calculate. Instead of finding what percentage 21 is of 14 (as was asked), they calculated the percentage 14 is of 21. The latter, with a result less than 100 percent, is the more frequent form of the question. Finally, on a geometry problem that prompted students to find the area of a figure, they operated on the values provided in the problem without regard to whether the values were the appropriate ones. They simply took familiar operations and applied them to what was there.

Students' tendencies to make the errors outlined above were quite consistent: when they could make these errors, they did. We looked at the ten items on each of the two tests that were answered correctly by the most students. None of these items provided opportunities for making the kinds of errors identified above.

Do We See Evidence of Reasoning?

As with many standardized mathematics tests, the items of the MDTP focus on procedural knowledge; very little reasoning is called for. Because of that, it is difficult to assess reasoning from test scores. When we examine frequent procedural errors though, we can see many cases where, had students reasoned at all about their answer choice, they wouldn't have made the error. This lack of reasoning was pervasive. It was apparent on both the Algebra Readiness and the Elementary Algebra tests, across math subtopics, and on both "easy" and "difficult" items. We will provide a number of specific examples.

On the Elementary Algebra test, students were asked to find the decimal equivalent of an improper fraction. Only one of the available answer options was greater than 1, yet nearly a third of students (32 percent) selected a wrong answer. If students had had a sense for the value of the improper fraction (simply that it represented a number greater than 1) and then scanned the options, they could have eliminated all four distractors immediately and without doing a calculation of any kind.

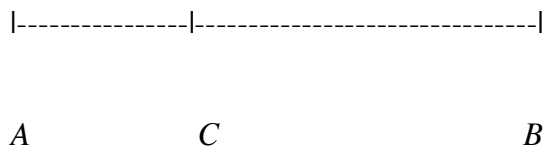
Another item prompted students to subtract a proper fraction (a value nearly 1) from an improper fraction (a familiar form of one and a half). Again, if students had examined the fractions and developed a sense of what the answer should be, they would have known that it would be slightly more than a half. Surprisingly, 13 percent of

students chose a negative number as their answer, revealing that they could not detect that the first fraction was greater than the second.

A geometry problem asked students to find one of the bases of a right triangle, given the lengths of the other two sides. Nearly a quarter of students selected an answer that was geometrically impossible. They selected lengths that could not have made a triangle, given the two lengths provided. Two of their answer choices yielded triangles with two sides whose sum was equal to the length of the third side. The third choice produced a triangle with a base longer than the hypotenuse.

Students were presented a problem that provided diagrams of similar triangles and asked to identify the length of one of the sides, but one of the answer options was strikingly out of range. The line segment AB in Figure 1 was provided as the base of the larger triangle.

Figure 1. Line segments AC and AB represent the bases of two similar triangles.



The length of $AB = 28$. Students were to use the values of the other two bases to find the length of AC . Thirteen percent of students said that the length of AC was 84. What they did was notice that one of the bases was three times the other and therefore multiplied 28 by 3 to get their answer. Presumably, they didn't check to see if their answer made logical sense.

So is it the case that students are incapable of reasoning? Are they lacking the skills necessary to estimate or check their answers? In at least one case, we have

evidence that community college students have the skills they need. On one Elementary Algebra test item, students were provided values for x and y and were asked to find the value of an expression in which they were used. (Though the expression included a fraction, there was no need for either simplification or division, two error-prone tasks.) The item proved to be the third easiest on the test, with nearly three quarters of students answering correctly. Their performance on the item demonstrates that they are capable of plugging in values and using basic operations to solve. That skill would have eliminated a great number of frequently chosen wrong answers *if students had thought to use it*. If students had only chosen a value for the variable and substituted this value into both the original expression and their answer choice, they could have caught the mistakes they'd made doing such things as executing operations and simplifying. Some people may think of plugging answer options into an item prompt as purely a test-taking strategy, but we argue that verification is a form of reasoning. In this case, it shows that the student knows the two expressions are meant to be equivalent, and should therefore have the same value.

We noted in the introduction that students are taught mathematics as a large number of apparently-unrelated procedures that must be memorized. It appears from the MDTP that the procedures are memorized in isolated contexts. The result is that a memorized procedure isn't necessarily called upon in a novel situation. Procedures aren't seen as flexible tools – tools useful not only in finding but also in checking answers. Further, what do students think they are doing when they simplify an algebraic expression, or for that matter simplify a fraction? Do they understand that they are

generating an equivalent expression or do they think they are merely carrying out a procedure from algebra class?

We cannot know from the MDTP the degree to which students are capable of reasoning, but we do know that their reasoning skills are being underutilized and that their test scores would be greatly improved if they had a disposition to reason.

Survey

Study Participants

Students were recruited from four community colleges in the Los Angeles metropolitan area. All were enrolled in 2009 summer session classes, and all were taking a developmental mathematics class. The breakdown of our sample by math class is shown in Table 4.

Table 4. Number of survey study participants, by class in which they were enrolled.

Class	N
Arithmetic	82
Pre-Algebra	334
Elementary Algebra	319
Missing Data	13

We collected no data from Intermediate Algebra students, even though it, too, is not a college-credit-bearing class. Our sample mainly lies in the two most common developmental placements: Pre-Algebra and Elementary Algebra.

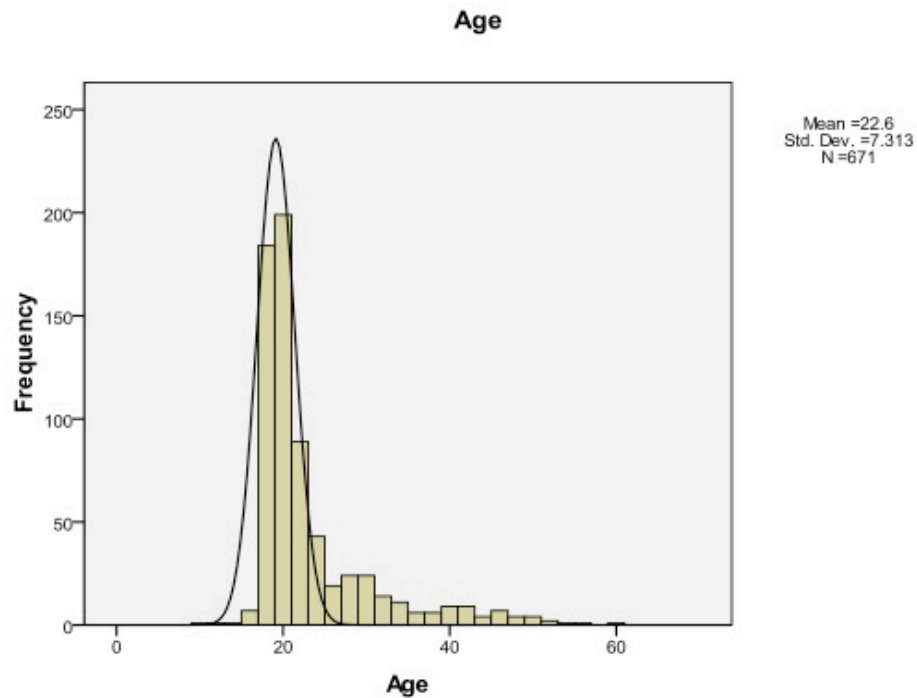
We asked students to tell us how long it had been since their last math class and the results are shown in Table 5.

Table 5. Length of time since survey study participants' most recent math class.

Time Since Last Math Class	N
1 year or less	346
2 years	118
3-5 years	83
More than 5 years	149
Missing Data	52

Although the modal student in our sample was 20 years old, it is evident in the histogram below (in Figure 2) that the age distribution has a rather long tail out to the right, with a number of students in their 30s and 40s.

Figure 2. Distribution of age of survey study participants.



Survey Items

To construct the survey, we began by listing key concepts in the mathematics curriculum, from arithmetic through elementary algebra. They included comparisons of fractions, placement of fractions on a number line, operations with fractions, equivalence of fractions/decimals/percents, ratio, evaluation of algebraic expressions, and graphing linear equations. Survey items were created to assess each of those concepts. To better understand students' thinking, several of the items included also the question, 'How do you know?'

The initial survey consisted of 12 questions divided into three forms of four questions each. Each student was randomly given one of the three forms.

Understanding of Numbers and Operations

The first items we will examine tried to get at students' basic understanding of numbers, operations and representations of numbers. We focused on fractions, decimals and percents.

In one question students were instructed: "Circle the numbers that are equivalent to 0.03. There is more than one correct response." The eight choices they were asked to evaluate are shown in Table 6, along with the percentage of students who selected each option. (The order of choices has been re-arranged according to their frequency of selection.)

Table 6. Survey question: Circle the Numbers Equivalent to 0.03.

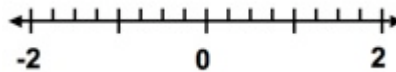
Response Option	Percent of Students Who Marked It as Equivalent to 0.03
3/100	67*
3%	53*
0.030	38*
3/10	23
0.30%	12
30/1000	9*
0.30	6
3/1000	3

*indicates a correct option

Only 4 percent of the students got all answers correct. The easiest two options ($3/100$ and 3%) were correctly identified by only 67 percent and 53 percent of the students, respectively. It appeared that as the answers departed further from the original form (0.03) students were less likely to see the equivalence. Interestingly, only 9 percent of students correctly identified $30/1000$ as equivalent, even though 38 percent correctly identified 0.030 . It appears that some students learned a rule (adding a zero to the end of a decimal doesn't change the value), yet only some of these saw that 0.030 was the same as $30/1000$. Students clearly are lacking a basic fluency with the representations of decimals, fractions and percents.

The students enrolled in Elementary Algebra did significantly better than those enrolled in Pre-Algebra or Arithmetic ($F(362, 2) = 5.056, p < .01$). Yet, even of the students in Algebra, only 17 percent correctly chose $30/1000$ as equivalent to 0.03 ($F(156, 2) = 7.290, p = .001$).

Another question asked students to mark the approximate position of two numbers (-0.7 and $1 \frac{3}{8}$) on this number line:



Only 21 percent of students were able to place both numbers correctly. 39 percent correctly placed -0.7 , and 32 percent, $1 \frac{3}{8}$. Algebra students performed significantly better than the Arithmetic students, but, only 30 percent of Algebra students marked both positions correctly.

On another question students were asked:

If n is a positive whole number, is the product $n \times 1/3$ greater than n , less than n , equal to n , or is it impossible to tell.

Only 30 percent of students selected the correct answer (i.e., less than n). Thirty-four percent said that the product would be greater than n (assuming, we think, that multiplication would always results in a larger number). Eleven percent said the product would be equal to n , and 26 percent said that they could not tell (presumably because they think it would depend on what value is assigned to n).

Interestingly, students in Algebra were no more successful on this question than were students in either of the other two classes ($F(176, 2) = 2.020, p < 0.136$). And, students who reported longer time since their last math class (i.e., 2 years ago) actually did better than students who had studied mathematics more recently (i.e., a year or less ago; $F(166, 3) = 3.139, p = 0.027$). This kind of question is not typical of what students would confront in a mathematics class; they are not asked to calculate anything, but just to think through what the answer might be. Perhaps the longer students have been away from formal mathematics classes, the less likely they are to remember what they are supposed to do, and the more they must rely on their own understanding to figure out how to answer a question like this one.

Do We See Evidence of Reasoning?

As we analyzed the students' responses, we started to feel that, first, students will whenever possible just fire off some procedure that they seem to have remembered from before, and, second, that they generally don't engage in reasoning at all, unless there is just no option. When they do reason they have difficulty. No doubt this is due in part to the fragile understanding of fundamental concepts that they bring to the task. It also

indicates a conception of what it means to do mathematics that is largely procedural, and thus a lack of experience reasoning about mathematical ideas.

We asked students:

Which is larger, $4/5$ or $5/8$? How do you know?

Seventy-one percent correctly selected $4/5$, 24 percent, $5/8$ (4 percent did not choose either answer).

Twenty-four percent of the students did not provide any answer to the question, "How do you know?" Those who did answer the question, for the most part, tried whatever procedure they could think of that could be done with two fractions. For example, students did everything from using division to convert the fraction to a decimal, to drawing a picture of the two fractions, to finding a common denominator. What was fascinating was that although any of these procedures could be used to help answer the question, students using the procedures were almost equally split between choosing $4/5$ or choosing $5/8$. This was often because they weren't able to carry out the procedure correctly, or because they weren't able to interpret the result of the procedure in relation to the question they were asked. Only 6 percent of the students produced an explanation that did not require execution of a procedure: they simply reasoned that $5/8$ is closer to half, and $4/5$ is closer to one. No one who reasoned in this way incorrectly chose $5/8$ as the larger number.

We asked a related question to a different group of students:

If a is a positive whole number, which is greater: $a/5$ or $a/8$?

If one is reasoning then this should be an easier question than the previous one. Yet, it proved harder, perhaps because many of the procedures students used to answer the previous question could not be immediately executed without having a value for a .

Only 53 percent of our sample correctly chose $a/5$ as the larger number. Twenty-five percent chose $a/8$, and 22 percent did not answer.

We followed up this question by asking, "How do you know?" This time, 36 percent were not able to answer this question. (Interestingly, this percentage was approximately the same for students who chose $a/5$ as for those who chose $a/8$.) Of those who did produce an answer, most could be divided into three categories.

Some students simply cited some single aspect of the two fractions as a sufficient explanation. For example, 5 percent simply said that "8 is bigger" or "8 is the larger number. All of these students incorrectly chose $a/8$ as the larger number. In a related explanation, 17 percent mentioned the denominator as being important - which it is, of course - but half of these students incorrectly chose $a/8$ as the larger number.

Another group of students (10 percent) used a procedure, something they had learned to do. For example, some of them substituted a number for a and then divided to find a decimal, but not always the correct decimal. Others cross multiplied, ending up with $8a$ and $5a$, or found a common denominator (40ths). Approximately half the students who executed one of these procedures chose $a/5$ as larger, and half chose $a/8$. They would execute a procedures, but had a hard time linking the procedure to the question they had been asked to answer.

The most successful students (15 percent) produced a more conceptual explanation. Some of these students interpreted the fractions as division. For example,

they pointed out that when you divide a number by five you get a larger number than if you divide it by eight. Others drew pictures, or talked about the number of "pieces" or "parts" a was divided into. Some said that if you "think about a pizza" cut into five pieces vs. eight pieces, the five pieces would be larger. Significantly, all of the students who used these more conceptual explanations correctly chose $a/5$ as the larger number.

Four percent of students said that it was impossible to know which fraction was larger "because we don't know what a is." We know from previous research that it is difficult for students to make the transition to algebra, to learn to think with variables about quantities. These results from much older students suggest that the lack of experience thinking algebraically may actually impede students' understanding of basic arithmetic.

Another test item that revealed students' ability to reason was the following:

If $a + b = c$, which of the following equations are also true? There may be more than one correct response.

The possible responses, together with the percentage of students who chose each response, are presented in Table 7.

Table 7. Response options and percent of students choosing them in answer to the question, "If $a + b = c$, which of the following equations are also true?"

Response Option	Percent Students Choosing
$b + a = c$	91*
$c = a + b$	89*

$c - b = a$	45*
$c - a = b$	41*
$b - c = a$	17
$a + b - c = 0$	28*
$c - a + b = 0$	9

*indicates a correct option

Most of the students knew that the first two options were equivalent to $a + b = c$. They knew that the order didn't matter ($a + b = b + a$) and they knew that you could switch what was on each side of the equals sign without affecting the truth of the equation. Still, 10 percent of students did not know these two things.

It proved much harder for students to recognize that if $a + b = c$, then $c - a$ would equal b (or, $c - b$ would equal a), with only 45 percent and 41 percent of the students choosing each of these options. Students could have arrived at these two answers either by executing a procedure (e.g., subtracting b from both sides of the equation) or by understanding the inverse relationship between addition and subtraction.

It is illuminating to look at the patterns of response students gave to the following three options:

$$c - a = b$$

$$c - b = a$$

$$b - c = a$$

Even though 40+ percent of students correctly chose the first two options, fully 13 percent chose all three options as correct. This finding suggests that students are examining each option in comparison to the original equation ($a + b = c$), but not

necessarily looking at the options compared with each other. It is hard to imagine how someone could believe that the latter two options are simultaneously true, unless they mistakenly think that the order of subtraction ($c - b$ vs. $b - c$) is not important, overgeneralizing the commutative property of addition to apply to subtraction, as well. Only 25 percent of the sample correctly chose both of the first two options but not the third.

A similar analysis can be done with the last two options:

$$a + b - c = 0$$

$$c - a + b = 0$$

Although 28 percent of the students correctly selected the first option as true, only 19 percent selected *only* the first option and *not* the second. Nine percent of the students selected both options as true. Interestingly, for both of these last two pattern analyses, there was no significant effect of which class students are in on their ability to produce the correct pattern of responses: Elementary Algebra students were no more successful than Pre-Algebra or Arithmetic students. This is a very intriguing result. It suggests that students who place into algebra may not really differ all that much in terms of their conceptual understanding from students placed into basic arithmetic or pre-algebra classes. The main difference may simply be in the ability to correctly remember and execute procedures, a kind of knowledge that is fragile without a deeper conceptual understanding of fundamental mathematical ideas.

In fact, none of the other items presented in this section ($4/5$ vs. $5/8$ or $a/5$ vs. $a/8$) showed significant differences in performance across the different classes. Clearly, there must be something different across these three classes of students - hence their placement

into the different classes, presumably based on performance on a placement test. Yet, in terms of reasoning and understanding in the context of non-standard questions, we could find few differences.

For the next two questions we told students the answer, but asked them to explain why it must be true.

Given that x is a real number, neither of these equations has a real solution. Can you explain why that would be the case? The equations were:

$$x + 1 = x$$

$$x^2 = -9$$

Forty-seven percent of the students could not think of any explanation for why there would be no real solution to the first equation. For the second equation, 50 percent could not generate an explanation. An additional 8 percent of students for the first equation (7 percent for the second equation) said that it would not be possible to know if the equations were true or not unless they could know what x is.

For the first equation, 23 percent of students tried to solve it with an algebraic manipulation. For example, they started with $x + 1 = x$, subtracted x from both sides, and then wrote down on their paper $1 = 0$. Or, they subtracted 1 from each side and wrote: $x = x - 1$. Once they had obtained these results they did not know what to do or say next. Similarly, for the second equation, 20 percent launched into an algebraic manipulation. Starting with $x^2 = -9$, for example, these students tried taking the square root of both sides, subtracting x from both sides, and so on.

Only 10 percent of students were able to give a good explanation for the first equation, and only 9 percent for the second equation. For the first equation, these correct

explanations included: "Because if you add 1 to anything or any number, the answer has to be different than the letter in the question or equation;" or, "x can't equal itself + 1."

For the second equation, correct explanations included: "A squared number should be positive since the first number was multiplied by itself;" or, "not possible because positive times positive will always be positive and negative times negative will always be positive."

Two more questions help to round out our exploration of students' reasoning about quantitative relations.

$$x - a = 0$$

Assuming a is positive, if a increases, x would:

- *increase*
- *decrease*
- *remain the same*
- *Can't tell*

Only 25 percent of students correctly chose increase. Thirty-four percent chose decrease, 23 percent, remain the same, and 11 percent said that you can't tell.

$$ax = 1$$

Assuming a is positive, if a increases, then x would:

- *increase*
- *decrease*
- *remain the same*
- *Can't tell*

Only 15 percent of students got this item correct (decrease). Thirty-two percent said increase, 33 percent, remain the same, and 14 percent said that you can't tell.

As with the previous items in this section, there was no significant difference in performance between students taking Arithmetic, Pre-Algebra and Algebra.

Interviews

The interview portion of our investigation sets out to address each of our three research questions. To reiterate, they are: 1) What do students understand about mathematics, 2) What does it mean to do mathematics, and 3) Can students reason if provided an opportunity and pressed to do it? We open the interview with questions about what it means to do mathematics, asking in a variety of ways what students think about the usefulness of math and what it takes to be good at it. That portion of the interview is followed by discussion centered around eight mathematical questions inspired by findings from the survey questions (previous section). For each we've anticipated possible responses and have created structured follow-ups. The general pattern is to begin each question at the most abstract level and to become progressively more concrete, especially if students struggle. Each of the eight questions concludes with prompts that press for reasoning. A copy of the complete interview protocol, along with annotations explaining what we were hoping to learn from each question, is attached to the end of this report (Appendix).

We are currently doing these interviews, and a full analysis of the results will be forthcoming. However, we will provide some of the interview responses from one of our early subjects, as it helps to fill in our picture of students' mathematical understandings.

Case Study: Roberto²

Roberto is in his first semester in community college this fall, having graduated from high school in the spring. He recently turned 18, and plans to become a history teacher. To reach his goal of becoming a teacher, he needs to eventually transfer to a four-year university. To do this, Roberto knows he must successfully make it through the sequence of developmental math courses that leads to finally taking the one required credit-bearing mathematics course.

Roberto enjoys math class when the teacher makes it challenging and interesting. For Roberto this means that the teacher challenges students, and makes students work hard until the math makes sense. He does not think math is “that hard,” and thinks he is good at math because he has a good memory. However, he does not believe that he has to remember everything. Rather, he might remember something that then “triggers a sequence” of steps. He is enrolled in Basic Arithmetic, the lowest level developmental mathematics class offered, and he knows this is the result of his performance on the college’s placement test for mathematics course enrollment. He attributes his performance to not remembering so well.

Roberto took Algebra 1 both as an 8th grader, and as a freshman and sophomore in high school. In high school, his class was a one-year course spread over four semesters. Despite this extensive experience in Algebra classes, he is still two semester-long courses away from the developmental course that is equivalent to a high-school level Algebra 1 course.

² Roberto is a pseudonym. To listen to his interview, go to <http://vimeo.com/7045271>.

For Roberto there seems to be a difference between doing in-school mathematics and out-of-school mathematics. Over the course of the interview, he asks repeatedly if he is allowed to solve a problem or answer a question a certain way. For example, when asked what number would fill in the blank to make the equation $7 + 5 = \underline{\quad} + 4$ true, Roberto explained that both sides would need to have the same value because of the equal sign. He asks if he can do it “just by looking at it, or by finding a way to do it?” When the interviewer says “It’s up to you,” Roberto chooses to do it “just by looking at it.” He says the left side is a total of 12, and so the right side must be 12, as well. For this reason 8 should fill in the blank. But, when asked immediately after about the value of x in the equation $7 + 5 = x + 4$, Roberto remarks “Well, if we’re talking algebra, you would subtract 4 and move it to this side.” He initially says x would be 12, but catches himself and says “I didn’t do it right.” He corrects his mistake, mentioning that he forgot to subtract 4 from 12, and agrees that no number other than 8 could be a solution for either equation.

When asked if he thinks it would be okay to think about the second equation the same way he thought about the first equation, Roberto shares that he thinks you should be allowed to if you can, but none of his teachers allowed him to. He says they thought he was cheating. If he were a teacher he would allow students to solve equations the way it makes sense to them.

Roberto’s understanding of the meaning of the equal sign in an equation is quite robust. When presented with other equations, such as $7 + 5 = x$, $\frac{1}{2} = \frac{2}{4}$, and $2 = 2$, Roberto maintained his conception of the equal sign as showing a balanced relationship. He offered that there are two other signs that show a relationship between numbers: the

'greater than' and 'less than' signs. But if there is an equal sign between the quantities, you must keep them equal.

Regarding the first research questions we set out to answer, Roberto understands the equal sign beyond what many students understand. Research shows that it is not uncommon for students to see the equal sign as a cue to "do something" like find an answer (Knuth, 2006). Not Roberto. He understands the equal sign as relational, and can use this relationship to solve equations. However, it is not clear if he sees algebraic manipulations such as "subtract 4 from each side" as a mechanism for maintaining the relationship.

In answer to our second question, Roberto knows he needs to do school mathematics to achieve his goals for higher education, but knows that there are other ways to think about mathematics and solve problems. It is striking that he was very concerned about equality in an equation with a blank space. But when the blank was replaced with x , he deferred to common algebraic procedures and by carrying out the procedure incorrectly he ruined the balance he said was so important.

As for the third question, can Roberto reason if provided the opportunity?

When Roberto is asked to tell if a is a positive whole number, whether $a/5$ or $a/8$ is larger, he replies that fifths are larger than eighths. He initially does not recognize that the numerator of each expression should be the same according to the expression given. Once this is cleared up, he explains that $a/5$ would always be larger because some number of fifths would be larger than that same number of eighths.

Roberto's confusion about whether the " a " has the same value in each fraction, and his differing approaches to solving the two equations above are evidence that there is

a disconnect between his mathematical understanding and his performance on mathematics tasks involving algebraic notation. The seriousness of this disconnect becomes more evident with his responses to questions about adding $1/3$ to a number and multiplying a number by $1/3$.

When posed with a question about making a comparison of a number (a) to the sum of the number and $1/3$ (x) using the equation $a + 1/3 = x$, Roberto responds that he needs to first find out either a or x . He sees the equation as something to solve rather than something he can use to reason about the relationship of the quantities in the equation. Roberto then responds that the sum will be smaller. This response is flawed for two reasons. First, Roberto explains that in order to add a whole number and a fraction, one needs to rewrite the whole number as a fraction. He says that if $a = 1$, he would need to rename it as “1 over 0”. Then, he adds the numerators (1 and 1) and the denominators (0 and 3) to get a sum of “2 over 3” or two-thirds.

Next, he compares the sum to $1/3$ rather than to the original number, and changes his answer to “larger.” This might easily be remedied by pointing out which two numbers should be compared (the original number and the sum). The interviewer chooses not to do this because of Roberto’s more serious mistake with the procedure. The interviewer does ask Roberto to try and add $2 + 1/3$ without changing 2 to a fraction. Roberto says that he was probably taught to do this, but doesn’t remember how. He says, “I was taught so many things.”

Roberto is then asked to think about multiplying a number by $1/3$ and whether the result would be smaller than the original number, larger than the original number, equal to the original number, or “can’t tell”. The equation $a * 1/3 = x$ is written for him.

Roberto proceeds to choose a number for a , and then to multiply the number by 1 and by 3 to get an equivalent fraction to $1/3$. He then simplifies this fraction and concludes that the result will be the same, again comparing the result to $1/3$ rather than the original number. In the middle of this process, Roberto tries to recall what he has been taught to do and comments that he has been “taught by like seven million teachers how to do this.” He uses the same process with $a = 3$, with the same result. The interviewer asks him to think about $1/2$ of a number without writing anything on the paper. Roberto talks about how finding half of a number is dividing the number by 2, and successfully finds $1/2$ of a few numbers. He correctly states that $1/2$ of a number will be smaller than the number you start with, and adds that it depends on if the number is negative or positive.

The interviewer asks if he could use that same process to find $1/3$ of a number. Roberto replies that taking $1/3$ of a number is like dividing the number into three equal parts. When the interviewer reminds Roberto that he earlier said that multiplying a number by $1/3$ gives a number equal to $1/3$, he says he doesn't even remember what he did with that problem. Clearly there is a disconnect between his ability to reason about $1/2$ of a number and $1/3$ of a number and thinking about either of these as multiplying by a fraction.

The implications of this disconnect and his difficulty with notation become clear when Roberto is asked to choose from among several expressions which could be used to find $1/2$ of a number. (The choices are shown in the Appendix.) Roberto chooses the expression with the exact wording “ $1/2$ of n ,” but also chooses two incorrect expressions. He chooses $n - 1/2$ and $n \div 1/2$, which are consistent with how he described his process for finding half of a number. In his mind, the result is smaller, and he had talked about both

subtraction and division. Plus, $1/2$ must be part of the expression even though he talked about dividing by 2. The expressions he chooses are the only two expressions that fit this criteria. Clearly Roberto is able to find half of positive even numbers, but he is unable to choose an algebraic expression that he could use to find half of a number. We must ask ourselves if lack of knowledge of notation is equivalent to not understanding a mathematical concept.

Throughout his interview, Roberto often attempted to explain his thinking process by referring to a number line even when the problem was not about a number line. Each time he did this he was correct. Also throughout the interview, Roberto mentioned trying to recall what he was supposed to do based on what he had been taught by “seven million teachers.” He claims that he is good at math because he has a good memory. Roberto does not have a very good memory. In fact, besides basic whole number calculations, he was incorrect with almost every calculation procedure he tried. Does this mean he does not understand mathematics or cannot reason about mathematics? Not necessarily.

Roberto invoked inverse operations, cleverly used number lines, and made generalizations. When asked to describe the equation $x - y = 0$, he used an analogy of prison, saying that 0 keeps x and y from being whatever they want to be. When the interviewer was able to give Roberto no other choice but to reason, he could do it. One example is Roberto’s interpretation of the relationship of x and y in the equation $x - y = 1$ as x is always one more than y . He also knew there would be infinitely many x, y pairs. Further, Roberto was willing to reason. He commented both midway through and at the end of the interview that he was having fun. In fact, this student who is spending a

semester in basic arithmetic was willing to be interviewed and share his thinking for nearly an hour and a half.

For Roberto mathematics is interesting and fun when he is being challenged to think. However, it is not clear what Roberto has been asked to think about in his many mathematics classes. He clearly has some understanding and the ability to reason, but one wonders if his shallow knowledge of procedures has been his downfall. What may at first appear to be gaps in understanding eventually reveal themselves to be gaps in procedural knowledge and notation, exacerbated by the disconnect between his often correct reasoning and his often incorrect procedures. Sadly, Roberto is not able to recognize this disconnect and too often defers to his memory. We should not expect otherwise, as he professed at the beginning of the interview that a good memory is the determining factor in being good at mathematics.

Conclusions

This was a small study in which we pieced together several pieces of data to paint a picture of what community college developmental mathematics students know about mathematics. The picture we paint is disturbing, and shows the long-term consequences of an almost exclusive focus on teaching mathematics as a large number of procedures that must be remembered, step-by-step, over time. As the number of procedures to be remembered grows – as it does through the K-12 curriculum – it becomes harder and harder for most students to remember them. Perhaps most disturbing is that the students in community college developmental mathematics courses did, for the most part, pass high school algebra. They were able, at one point, to remember enough to pass the tests they were given in high school. But as they move into community college, many of the

procedures are forgotten, or partly forgotten, and the fragile nature of their knowledge is revealed. Because the procedures were never connected with conceptual understanding of fundamental mathematics concepts, they have little to fall back on when the procedures fade.

It is clear from the interviews that students conceive of mathematics as a bunch of procedures, and one often gets the sense that they might even believe it is inappropriate to use reason when memory of procedures fails. Roberto, in our case study, asked at one point: 'Am I supposed to do it the math way, or just do what makes sense (paraphrased)?' He appears to think that the two are mutually exclusive. Roberto, remember, had taken elementary algebra three times in K-12: once in eighth grade, and then again for two years in ninth and tenth grades. He showed signs of being able to reason, but didn't bring reason to bear when his procedures were not working, nor was he able to notice that his answers resulting from procedures did not necessarily match his answers resulting from reasoning. He, like most of the students in this study, looked at each problem, tried to remember some procedure that could be applied to the problem (e.g., cross-multiply), and then tried to execute the procedure. Unfortunately, much of the time, either the procedure was not the correct one, or it was executed incorrectly, which led to the high incidence of mathematical errors.

The placement tests provide ample evidence that students entering community colleges have difficulty with the procedures of mathematics. What is clear from our data is that the reason for these procedural difficulties can be tied to a condition we are calling conceptual atrophy: students enter school with basic intuitive ideas about mathematics. They know, for example, that when you combine two quantities you get a larger quantity,

that when you take half of something you get a smaller quantity. But because our educational practices have not tried to connect these intuitive ideas to mathematical notation and mathematical procedures, the willingness and ability to bring reason to bear on mathematical problems lies dormant. The fact that the community college students have so much difficulty with mathematical notation is significant, for mathematical notation plays a major part in mathematical reasoning. Because these students have not been asked to reason, they also have not needed the rigors of mathematical notation, and so have not learned it.

But there also is some good news. In every interview that we have done so far, we have found that it is possible to coax the students into reasoning, first, by giving them permission to reason (instead of doing it the way they were taught), and second, by asking them questions that could be answered by reasoning. Furthermore, the students we are interviewing uniformly find the interview interesting, even after spending well over an hour with the interviewer thinking hard about fundamental mathematics concepts. This gives us further cause to believe that developmental math students might respond well to a reason-focused mathematics class in which they are given opportunities to reason, and tools to support their reasoning.

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