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Mathematics Education Dialogues

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All-New Forum Makes Debut!

Perhaps the major development in mathematics education in the twentieth century has been the transformation from mathematics beyond arithmetic as a subject expected to be learned only by some to a subject to be learned by all. At the same time, within the past generation, the learning of traditional arithmetic itself has been placed into question by the existence of hand-held calculators, and calculators and computers have changed the ways in which many adults work with mathematics.

Simultaneous with these developments, and perhaps because of them, the learning of mathematics has emerged as a high-priority educational issue. Teachers are being expected to bring all students to levels of performance if not higher than those expected in previous generations, different from those expectations. Tests that originated in many of our states and provinces to monitor the performance of students and schools are increasingly being adapted or replaced by tests whose passage is required for graduation.

Many questions naturally emerge from these changes. Here are just a few:

- What mathematics should all students be expected to learn? For how long in their schooling should all students be expected to learn the same mathematics in the same classes? When and how should the differentiation take place?
- Should four-function calculators replace any of the paper-and-pencil skills of arithmetic? Do they require that new ideas be taught? How, if at all, should symbol-manipulating calculators affect the paper-and-pencil skills traditionally taught in algebra?
- Is it wise for high-stakes tests to be used to drive the mathematics curriculum? If so, what is the best way for this outcome to be accomplished? If not, what are the alternatives? When are we using too many tests?
- To what extent do different students learn in different ways? And if they do,

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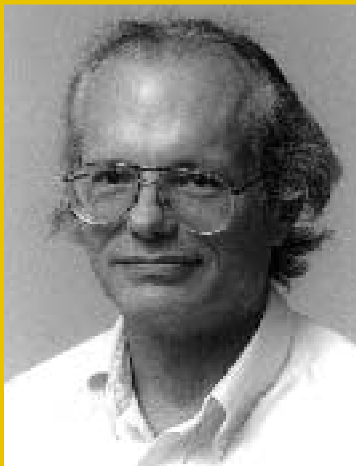
The purpose of Mathematics Education Dialogues is to provide a forum through which NCTM members can be well informed about compelling, complex, timely issues that transcend grade levels in mathematics education.

Inside this issue …



Underwood Dudley
(dudley@depauw.edu)

Underwood Dudley taught his first calculus course in 1957 and is amazed that after all the years he has been teaching them, students are still making the same mistakes. He hopes to see the new millennium in at DePauw University, here he has taught for 19% of the institution's existence. His last book, *Numerology*, was published by the MAA in 1997. Woody is the editor-elect of the *CMJ* (the *College Mathematics Journal*).

Is Mathematics Necessary?

Underwood Dudley
DePauw University

Is mathematics necessary? Necessary, that is, for citizens of the United States to function in the world of work? You would get that impression from reading various recent documents, some coming from high and official places. For example, *Moving Beyond Myths,* published by the National Academy of Sciences, says so [**5**, p. 11]:

> ***Myth:*** Most jobs require little mathematics.
> ***Reality:*** The truth is just the opposite: more and more jobs—especially those involving the use of computers—require the ability to use quantitative skills. Although a working knowledge of arithmetic may have sufficed for jobs of the past, it is clearly not enough for today, for the next decade, or the next century.

The anonymous author of that item presumably had mathematical training and thus should know that theorems are not proved by assertion. But if we look in the document for evidence for that supposed reality, we look in vain, so an assertion is all that it is.

Here is an excerpt from *Everybody Counts* [**6**, p. 4], written anonymously for the National Research Council. (Do documents issuing from important national organizations gain more weight when their authors are not identified? Some members of important national organizations evidently think so.) This report too says that mathematics is a vocational necessity:

> Just because students do not use algebra anywhere except in algebra class does not mean that they will not need mathematics in the future. Over 75 percent of all jobs require proficiency in simple algebra and geometry, either as a prerequisite to a training program or as part of a licensure examination.

A quick reading of that passage might leave the impression that algebra and geometry are used in 75% of *all* jobs, but that is silly. Just look at the next eight workers that you see and ask yourself if at least six of them require proficiency in algebra to do their jobs. (If you are a teacher of mathematics, it is not fair to look at eight colleagues.)

The anonymous author was careful to

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(This article is reprinted with the permission of the author and the Mathematical Association of America. The article appears in the November 1997 issue of the College Mathematics Journal.*)*

EXECUTIVE SUMMARY OF MATHEMATICS EQUALS OPPORTUNITY

RICHARD W. RILEY

Dick Riley was governor of South Carolina from 1978 to 1986 and served in the state legislature prior to that. His goals for education include voluntary national tests to ensure that all students master the basics of reading and mathematics, and organizing one million volunteer reading tutors across the nation.

Richard W. Riley
U.S. SECRETARY OF EDUCATION

In the United States today, mastering mathematics has become more important than ever. Students with a strong grasp of mathematics have an advantage in academics and in the job market. The 8th grade is a critical point in mathematics education. Achievement at that stage clears the way for students to take rigorous high school mathematics and science courses—keys to college entrance and success in the labor force. However, most 8th and 9th graders lag so far behind in their course taking that getting on the road to college is a long way off.

This report highlights the following findings:

• **Students who take rigorous mathematics and science courses are much more likely to go to college than those who do not.** Data from the National Educational Longitudinal Study (NELS) reveal that 83 percent of students who took algebra I and geometry went on to college within two years of their scheduled high school graduation. Only 36 percent of students who did not take algebra I and geometry courses went to college. While nearly 89 percent of students who took chemistry in high school went to college, only 43 percent of students who did not take chemistry went to college.

• **Algebra is the "gateway" to advanced mathematics and science in high school, yet most students do not take it in middle school.** Students who study algebra in middle school and who plan to take advanced mathematics and science courses in high school have an advantage: approximately 60 percent of the students who took calculus in high school had taken algebra in the 8th grade. However, 1996 NAEP data reveal that only 25 percent of U.S. 8th graders enrolled in algebra, and that low-income and minority students

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(The essay above is the Executive Summary of a white paper, Mathematics Equals Opportunity, prepared for U. S. Secretary of Education Richard W. Riley, 20 October, 1997.)

Making fun of "applications" … is as easy as swatting mosquitoes in a swamp in midsummer, and as useful.

If we can't be realistic, we can at least be honest.

Is Mathematics Necessary?

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add the qualification that the algebra and geometry may be necessary only for training or licensing. But I find even this claim, another bald assertion with nothing to back it up, unbelievable. Over 75% of all jobs? Incredible! I cannot imagine how that wildly inflated percentage was arrived at, unless the author was including having a high school diploma under "licensure examination." Authors who want to indulge in unsubstantiated percentages should be careful to have them consonant with common sense. For example, "99% of the mathematics done by the average person relates to money" [**2**]—now *that* I can believe.

Almost all jobs, I counter-assert, require no knowledge of algebra and geometry at all. You need none to be President of the United States, none to be a clerk at Wal-Mart, none to be a professor of philosophy, … the list extends indefinitely. Few jobs require knowing any mathematics beyond algebra. You might think that engineers, of all people, would need and use calculus, but this seems not to be so [**7**]:

> Why do 50% (probably closer to 70%) of engineers and science practitioners seldom, if ever, use mathematics above the elementary algebra/trigonometry level in their daily practice?
>
> My work has brought me into contact with thousands of engineers, but at this moment I cannot recall, on average, more than three out of ten who were well versed enough in calculus and ordinary differential equations to use either in their daily work.

If 70% of *engineers* don't need calculus to do their jobs, then how many of the 500,000 or so students that we put through calculus every year will? Minutely few, so we should not tell them how tremendously *useful* calculus is going to be to them when they go to work. If most engineers can do quite well with only algebra and trigonometry (or perhaps even less), is it not reasonable that nonengineers can survive and flourish with arithmetic, or even less? Yes, it is.

Were algebra necessary for 75 percent of all jobs, our algebra textbooks would be filled with on-the-job problems, since examples would be so plentiful. But they are not. Open any textbook at random—I will open a new one, just published—and what you find are problems like this:

> Through experience and analysis, the manager of a storage facility has determined that the function $s(t) = -3t^2 + 12t + 10$ models the approximate amount of product left in the inventory after t days from the last resupply. We want to find when the supply of the product will be exhausted and a new resupply needed.

Real inventories do not behave this way. For one thing, they do not *increase* after the resupply, from 10 at $t = 0$ to 22 at $t = 2$. For another, they usually decrease linearly, not quadratically. Besides, I doubt that warehouse managers, even 75% of them, use formulas to decide when to reorder.

Making fun of the "applications" that appear in textbooks is as easy as swatting mosquitoes in a swamp in midsummer, and as useful. What such problems actually illustrate is that the mathematics in the textbooks has no application to the world of warehouses and work. Does this mean that we should teach less mathematics? No; we should teach more. *Everyone* should learn algebra, but not because it is necessary for managing warehouses. Does it mean that we should stop assigning "applied" problems? Certainly not; we should assign more. Problems expressed in words are the best kind, but they should all start with "Suppose that…." If we can't be realistic, we can at least be honest.

Those who know not history …

Let us look at history. Those ignorant of history too often assume, knowing no better, that the world has always been much as it is now, which is seldom so. Today, with near-universal instruction in arithmetic and algebra, it is easy to suppose the curriculum has always been like that. But it has not. Algebra was not always taught to everyone. Not only that, even arithmetic itself is a relative newcomer.

Here is a report from Massachusetts in the early 1800s. Not the 1700s, nor the 1600s, the 1800s [**3**, p. 13]:

> Until within a few years no studies have been permitted in the day school but spelling, reading and writing. Arithmetic was taught by a few instructors one or two evenings a week. But in spite of the most determined opposition, arithmetic is now being permitted in the day school.

Opposition to arithmetic! How could anyone possibly be opposed to arithmetic? It is difficult for us to imagine.

The explanation is that arithmetic was a vulgar subject. As Patricia Cline Cohen tells us in *A Calculating People: The Spread of Numeracy in Early America,* a book that deserves to be more widely known [**1**, p. 26]:

> Those of high social rank, theoretically above the world of getting and spending, did not deign to study the subject. The most respectable English public schools, like Eton and Harrow, did not offer any instruction in arithmetic until well into the nineteenth century.

The English attitude was exported to the colonies [**1**, p. 49]:

> The founding generation arrived in Massachusetts in the 1630s with the highest number of university degrees and the highest rate of literacy of any migratory group. Within a decade they instructed towns to establish local grammar schools and had set up Harvard College to provide high-level training for homegrown ministers. But arithmetic was not among the subjects considered basic for Puritan children to learn.

Nevertheless, the colonies, and England, not only survived but thrived, economically as well as culturally. Some people believe that the eighteenth century represented a peak of civilization from which we have declined. I would not go that far, and I much prefer living in our times, with its plumbing and penicillin, computers and compact disks, anesthesia and even its automobiles, yet history clearly shows that arithmetic in the schools is not needed for a high civilization. How can that be? Easily enough: workers learn what they need *on the job.* What happens in the schools simply does not matter.

Here is a report on the situation in Boston in 1789 [**1**, p. 131]. See if it does not sound familiar today:

> [There was a requirement] that boys aged eleven to fourteen were to learn a standardized course of arithmetic through fractions. Prior to this act, arithmetic had not been required in the Boston schools at all. Within a few years a group of Boston businessmen protested to the School Committee that the pupils taught by the method of arithmetic instruction then in use were totally unprepared for business. Unfortunately, the educators in this case insisted that they were doing an adequate job and refused to make changes in their programs.

Of course the students were unprepared for business, one reason being that it is neither wise nor practicable to try to prepare all students for all possible jobs. Another is that the "applications" in school books were just as phony as ours [**1**, p. 122]:

> Here is a typical word problem, typical in its complexity and in its use of current events to suggest the utility of arithmetic:
>
> > Suppose General Washington had 800 men and was supplied with provision for but two months. How many of his men must leave him, that his provision may serve the remaining five months?
>
> In this particular case the student mechanically applied the Rule of Three, writing 2 : 800 :: 5 and then dividing 5 into 2×800 to get a final answer of 320. Now, 320 is the number of men who can be fed for five months, not the number who must leave. So Washington's troops would have gone hungry if the schoolboy or his master had been in charge of provisioning.

As Professor Cohen pointed out, if Washington ran short of provisions, he would try to get more instead of telling part of his army to go away.

The conclusion cannot be avoided that school mathematics is not now, and never has been, necessary for jobs. There are a few exceptions, of course, most being for the jobs of teaching the subject. And of course science—both physical and social—cannot advance without a supply of scientists able to use mathematics. But most of these people did not need to be bullied or cajoled into learning the subject.

Even more advanced mathematics turns out to be all too often not needed for work [**8**]:

> Presumably, with degrees in mathematics and statistics [students with mathematical majors] could pursue careers in their disciplines. But, for mathematicians and statisticians who would seek employment in commerce, i.e. in business, industry, or government, this presumption is not presently valid. In fact most, if not virtually all, such mathematical scientists currently employed in commerce do not work in their fields of expertise.

This holds even for those with higher degrees: the National Research Council "reports that at least 90 percent of nonacademically employed mathematical scientists who received master's degrees in 1986 do not work as mathematical scientists" [**8**].

A few years ago I heard an interesting talk at an MAA section meeting on the use of mathematics by employees of the

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Some people believe that the eighteenth century represented a peak of civilization from which we have declined.

How could anyone possibly be opposed to arithmetic?

Is Mathematics Necessary?

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Florida Department of Transportation. The department needs to calculate many things, including areas, and its method of finding the area of irregular shapes was surprising to me. When I asked the speaker how the department copes with new workers with varying degrees of mathematical training, the answer was that it doesn't: it had found that the only safe assumption is that new workers know nothing about mathematics, so they are taught what they need as it is needed. This is satisfactory to everyone. It does not imply that the time that new employees had spent in school trying to do problems in arithmetic, algebra, and geometry was wasted, but it had nothing to do with their jobs. Boston, 1789, Florida, 1993: some things do not change.

A way of thought.

Despite the initial opposition and continued irrelevance to jobs, mathematics instruction spread in the United States in the nineteenth and twentieth centuries. As the *History of Mathematics Education* [**3**] tells us, Harvard in 1816 required "the whole of arithmetic" for entrance. Until then addition, subtraction, multiplication, division, and the Rule of Three had been enough. After 1865, geometry was required as well. As the country was settled, secondary education expanded, and arithmetic moved from the academies and high schools to become an elementary school subject by the end of the nineteenth century [**3**, p. 27]. Algebra was an optional subject in some high schools, and it became possible to study calculus in the upper reaches of some colleges. Today years and years of mathematics is compulsory for all and calculus has become a high school subject.

How come? Because parents, school boards, society as a whole think that mathematics instruction is worth doing. On account of applications and jobs? Certainly not. The reason, I think, is that one of the tasks of schools is to do their best to teach students to think, and of all subjects none is better suited to this than mathematics. In no other subject is it so clear that reasoning can get results that are right, verifiably right. When you solve $x^2 + x = 132$ and get $x = 11$, you can then calculate $11^2 + 11$ and know that you are correct. No other subject has this capacity at the elementary levels. Mathematics increases the ability to reason and shows its power, all at the same time.

It is not fashionable these days to assert that mathematical training strengthens the mind, perhaps because that proposition is as impossible to prove as the proposition that music and art broaden and enrich the soul. But it is still believed by many people, including me. Some of our forebears had more confidence, as did John Arbuthnot (1667–1735) whose *On the Usefulness of Mathematical Learning* (c. 1700) proclaimed: "The mathematics are the friends of religion, inasmuch as they charm the passions, restrain the impetuosity of the imagination, and purge the mind of error and prejudice" [**4**, p. 70]. Even better, "[M]athematical knowledge adds vigour to the mind, frees it from prejudice, credulity, and superstition" [**4**, p. 67]. Though we no longer say such things out loud, the belief that they hold quite a bit of truth goes a long way toward explaining why people have supported and continue to support the mass teaching of mathematics, though many of them did not enjoy the experience when they underwent it.

Once a graduate of my school, a mathematics major, came back to campus to visit. I said to him, after finding out that his job was running a television station in Knoxville, Tennessee, "Well, I guess all that mathematics you learned hasn't been very useful." "Oh no," he replied, "I use it every day." I found this claim incredible (soap operas have no partial derivatives), so I pressed him. It turned out that he meant that he believed he used *the mathematical way of thinking* every day.

That is impossible to quantify and impossible to prove, but we cannot tell him that he is wrong. Nor should we.

It is time to stop claiming that mathematics is necessary for jobs. It is time to stop asserting that students must master algebra to be able to solve problems that arise every day, at home or at work. It is time to stop telling students that the main reason they should learn mathematics is that it has applications.

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… the only safe assumption is that new workers know nothing about mathematics, so they are taught what they need as it is needed.

Can you recall why you fell in love with mathematics?

Mathematics Equals Opportunity

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were even less likely to take algebra in the 8th grade.

• **Taking rigorous mathematics and science courses in high school appears to be especially important for low-income students.** Low-income students who took algebra I and geometry were almost three times as likely to attend college as those who did not. While 71 percent of those who took algebra I and geometry went to college, only 27 percent who did not take those courses went on to college. By way of comparison, 94 percent of students from high-income families, and 84 percent of students from middle-income families who took algebra I and geometry in high school went on to college. Sixty percent of students from high-income families and 44 percent of students from middle-income families who did not take algebra I and geometry went to college.

• **Despite the importance of low-income students taking rigorous mathematics and science courses, these students are less likely to take them.** Students from higher-income families are almost twice as likely as lower-income students to take algebra in middle school and geometry in high school. They are more than twice as likely to take chemistry.

Other important findings include:

• **Mathematics achievement depends on the courses a student takes, not the type of school the student attends.** Students in public and private schools who took the same rigorous mathematics courses were equally likely to score at the highest level on the NELS 12th grade mathematics achievement test.

• **Students whose parents are involved in their school work are more likely to take challenging mathematics courses early.** Students whose parents were involved in their education were more likely to take courses like algebra and geometry in the 8th and 9th grade than students whose parents were not involved.

• **The results of the Third International Mathematics and Science Study (TIMSS) reveal that the middle school mathematics curriculum may be a weak link in the U.S. education svstem.** While U.S. 4th graders scored above the international average in mathematics and science, U.S. 8th graders scored below average in mathematics, and only slightly above the international average in science. Initial analysis of TIMSS data also shows that the middle school mathematics curriculum in the U.S. is less challenging than in other countries. The curriculum of average 8th-grade mathematics classrooms in the U.S. resembles 7th grade curriculum elsewhere. Although algebra and geometry are integral elements of the middle school curriculum in other countries, only a small fraction of U.S. middle schools offer their students these topics.

Algebra in the Curriculum

Making a successful transition from arithmetic to more advanced mathematics, including algebra and geometry, has often been difficult for students. As a result, many mathematics programs in the U. S. are now systematically incorporating some fundamentals of algebra and geometry into the upper elementary grade curriculum. In these programs, 5th, 6th and 7th grade students are representing and solving equations, characterizing patterns and rates of change among variables, and using other fundamental algebraic concepts.

In addition, some middle and high schools are taking a new approach to advanced topics. While many schools offer the traditional model of separate courses for pre-Algebra, Algebra I, Geometry, Algebra II, Trigonometry, pre-Calculus and Calculus, these schools are integrating them. This approach is consistent with practices in other industrialized nations, which integrate algebra, geometry, and other topics throughout the elementary, middle, and high school years and offer a significant component of algebra in the 8th grade. Building a firm foundation in algebra during the elementary and middle school years eases the shift from arithmetic to advanced topics, whatever the format of students' new curriculum. NELS and NAEP, the two sources of national mathematics course-taking data analyzed in this brief, employ traditional courses titles, such as "algebra I" and "geometry." Thus, these titles are used throughout the brief.

Students whose parents are involved in their school work are more likely to take challenging mathematics courses early.

Many mathematics programs are now incorporating fundamentals of algebra and geometry into the upper elementary grade curriculum

Reactions to

"Is Mathematics Necessary?" and "Mathematics Equals Opportunity"

A Classroom Teacher Speaks Out

by Clay Burkett

Underwood Dudley has hit the nail on the head in his article "Is Mathematics Necessary?" The mathematics that is really necessary for the average citizen is arithmetic—the mathematics used by the masses to get by in everyday life. I love the point made that "were algebra necessary …, our algebra textbooks would be filled with on-the-job problems, since examples would be so plentiful." Most of the application problems I find in traditional textbooks are contrived at best and do nothing to convince the students of the relevance of mathematics.

However, Mr. Dudley's conclusion that mathematics transcends jobs because of the elegance and beauty that lies within does little to assist the classroom teacher. The elegance and beauty of mathematics do not intrigue the vast

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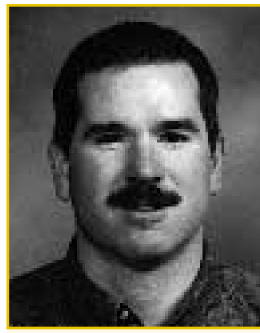

Clay Burkett

Clay Burkett has been involved in mathematics education for nine years, teaching everything from Saxon to SIMMS (Systemic Initiative for Montana Mathematics and Science). One year of his experience was spent writing curriculum for the SIMMS and STEM (Sixth through Eighth Mathematics) projects. He enjoys spending time with his wife and three children in the great outdoors of Montana.

Roger Howe

Roger Howe did graduate study under Calvin Moore at the University of California at Berkeley. He received his Ph.D. in 1969 and has been at Yale since 1974. He studies the implications of symmetry, especially the theory of group representations. In the 1990s, he has become more and more involved in mathematics education at all levels.

Mathematics as Externality: Implications for Education

by Roger Howe

Underwood Dudley sees mathematics as a sharpener of minds, and underscores its beauty. Secretary Riley tells students that mathematics is a key credential. Two justifications for mathematics could hardly be more different, but strictly speaking, they are not in conflict. Secretary Riley does not parry, but neatly sidesteps, Professor Dudley's thrust. He never claims, as do the NRC documents that Dudley so deftly skewers, that mathematics is necessary for careers. He only claims that it is necessary, or almost so, for college attendance. His figures are not round, and they are differentiated by ethnicity. They have an a priori plausibility that the NRC's claims lack.

What Secretary Riley is giving is the bleakest reason imaginable to study mathematics: no beauty and no utility—just a

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Merecedes McGowen

Mercedes McGowen, mmcgowen@harper.cc.il.us, teaches mathematics at William Rainey Harper College, Palatine, Illinois, a two-year college in the northwest suburbs of Chicago. Previously, she taught mathematics at Canton Junior High School and Elgin High School in Illinois. Her interests include the development of curricular materials based on research about how students learn mathematics.

The Splintered Vision: Wayfarers in Search of Different Roads.

by Merecedes McGowen

> The wayfarer,
> Perceiving the pathway to truth,
> Was struck with astonishment.
> It was thickly grown with weeds….
> Later he saw that each weed
> Was a singular knife.
> "Well," he mumbled at last,
> "Doubtless there are other roads."
>
> — Stephen Crane
> *The Wayfarer*

More than one-third of the 3.27 million undergraduate students enrolled in mathematics courses at two- and-four year colleges and universities in 1995 were enrolled in

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A Concerned Citizen's Perspective

by Penny Noyce

Underwood Dudley asks, "Is Mathematics Necessary?" and assures us that for the vast majority of jobs, calculus, geometry, and even algebra are not. He's probably right. For entry into medical school, I was required to take calculus; but I never needed calculus for learning anatomy or pharmacology, and I have certainly never used it in patient care. Calculus was a rite of passage, perhaps a weeding-out mechanism. What I do use, both in reading the medical literature and in thinking about patients, is probability.

Similarly, several years ago, the UNUM insurance company surveyed what mathematics the members of its work force actually required for their work. The findings? No algebra was necessary, but probability and statistics were vital.

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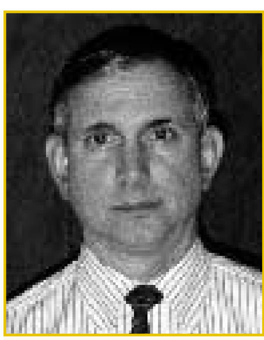
Penny Noyce

Trained as a physician with a specialty in internal medicine, Penny Noyce now spends most of her time working on issues of K–12 public education reform as a trustee of the Noyce Foundation. The Foundation focuses on the core academic areas of literacy, mathematics, and science. For the past three years, Noyce has served as coprincipal investigator of PALMS, the Massachusetts State Systemic Initiative.



John C. Souders Jr.

John C. Souders Jr. is vice-president for curriculum materials with CORD (Center for Occupational Research and Development). At CORD he developed the new textbook *CORD Algebra* and four new units in the *Applied Mathematics* Series that deal with higher order geometry topics. Previously Souders taught at the United States Air Force Academy and worked in nuclear engineering.

Reaction of a Curriculum Developer

by John C. Souders Jr.

Could two more diametrically opposed pieces of writing be found? In one, mathematics is presented as unnecessary for success in the world of work. In the other, mathematics is seen as the "gatekeeper" for future academic and career success. I'm not surprised that such a wide spectrum of opinion exists. After all, the teaching and learning of mathematics have been controversial for decades. I compliment the authors. Both present substantive and thought-provoking arguments.

The rallying call of Dudley's article could well have been the often-asked student question "Why do I have to learn this?" According to the article, a mastery of arithmetic is sufficient for

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Reactions—Clay Burkett

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majority of my students. They are more interested in the elegance and beauty of the opposite sex, or music, or movies, or sports …. I recently came across a profound statement that reflects our culture and hence the attitudes of our students. My paraphrase of this statement is, We in America like to think that we value art and education, but what we really value are sports and entertainment. Students want to know how and where mathematics is applicable to their lives. Relevance will motivate them to learn. We can get them to pass our classes through other means, but let's not confuse a grade with learning and valuing mathematics.

What about college-bound students? Are they not motivated to learn mathematics? Or, as the paper "Mathematics Equals Opportunity" suggests, does taking rigorous mathematics courses spur students on to attend college? It does not surprise me that students who take rigorous mathematics courses in high school tend to attend college in greater numbers than students who do not. However, any half-decent mathematician knows the difference between correlation and causation. After all, don't many colleges require these types of mathematics classes as prerequisites to admission? The thinking behind this paper appears to be that of mathematics as a pump—propelling students on to achievement in college and beyond. I like this idea and find it somewhat valid, yet to some degree mathematics will also act as a filter—separating the less motivated, less disciplined, and yes, even the less able.

As for the idea that college-bound students are motivated to learn mathematics, I submit the following informal survey. As a quick background to the survey, I currently teach Honors Geometry and Integrated Math IV, both of which meet college-entrance requirements. I had my students read the two articles of interest here and then held a discussion of the ideas contained therein. Many of my students believed that their high school mathematics courses were of no relevance to their lives or future careers—but merely hoops that they must jump through. When asked for a show of hands, thirty-one of forty-two Honors Geometry and seventeen of forty-one Integrated Math IV students indicated agreement.

One thing that caught my attention was that a greater proportion of students in the Integrated Math IV class thought that the mathematics they were learning was relevant to their lives in preparing them for college or a career. This fact did not come as a surprise to me, as the curriculum used in the Integrated Math IV class is one that is based on problem solving and applications. These students tend to see the relevance of mathematics more clearly because it is taught in a context of how and where the mathematics is used.

In conclusion I find some merit and some fault in both articles. Mathematics is not necessary on the one hand; and yet on the other, mathematics equals opportunity (has anyone been denied a job or college entrance for knowing too much mathematics?) for college and beyond. To me these articles point out the need for mathematics reform. Will we continue on with a mathematics curriculum that the TIMSS report describes as "a mile wide and an inch deep," or will we work to develop a curriculum that is meaningful and relevant to our students?

Reactions—Roger Howe

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credential, a hoop to jump through. However, in a sense its bleakness is a strength. It gets us an audience with no promises made except for the carrot of college attendance. If we then can help some member of this audience to sharpen their thinking habits, and to see the beauty we know in mathematics, then they have gained a precious treasure, and so have we.

Why do we have this problem? Why is it necessary to "justify" mathematics to students. And what, exactly, is wrong with the usefulness justification? Is mathematics not useful?

On the contrary, mathematics is more than useful; it is necessary, but not for (most) individuals. This is a great paradox of mathematics education: although our society is utterly dependent on mathematics for many of our daily needs, and even for the very shape of our civilization, for the most part we do not need personally to be able to master very much of the mathematics that serves us. It is built into our products, encoded in our practices, or available at a fee. It is the enduring luck of the species, and the sometimes bane of the profession, that mathematics is indestructible, infinitely recyclable, and totally fungible. A little goes a long, long way.

In *Quantum* magazine, an excellent publication, I read a wonderful detective story starring Johannes Kepler, the man who also discovered the laws of planetary motion. Kepler was in Linz, Austria, to get married. He needed wine for the wedding party. He was intrigued to see that the wine merchants of Linz computed the capacity of their barrels by making a single measurement, diagonally from the bunghole in the middle of the side, to the top of the far wall. Barrels are not uniform in shape or size, so how could a single measurement provide an adequate estimate of the volume? This simple method was a trade secret of the Linz Coopers (Barrelmakers) Guild. Kepler wanted to understand their secret. He modeled the barrels as two truncated cones joined end to end, and reasoned that the barrels must be made to have approximately the shape that would make the volume as large as possible for the given measurement. Maximizing the volume would provide two benefits at once. First, the vintners could charge the maximum amount for the given measurement. Second, because a maximum is also a stationary point, the volume will vary very little if the barrel differs slightly from the optimal shape, so customers will accept this method of calculation. After doing a little calculus (actually, precalculus, since calculus had not yet been invented!), Kepler concluded that the barrels should be cylindrical, with a ratio of height to diameter of $\sqrt{2}$. Allowing for the realities of manufacture

Reactions—Roger Howe

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(nonzero thickness of the barrel top, etc.), one finds that 1.5 is a sufficiently accurate, numerically manageable proxy. Once the shape is known, translating the length measurement into volume is a simple calculation, and the volumes for given lengths can be inscribed on the ruler. Having deduced this fact, Kepler sought an interview with the Elder of the Coopers Guild, to whom he revealed that he had discovered their secrets of barrel construction and measurement. The amazed Elder acknowledged that Kepler was correct, and said that these were trade practices handed down through the generations since the days of the Venerable Cooper. Here was a sophisticated and very useful piece of mathematics, embedded as a simple rule of thumb in the cooper's trade. It took a genius to reverse engineer it, and undoubtedly also, a genius to invent it, but it could be learned by rote by any apprentice cooper.

A very large part of the mathematics that we use is like the cooper's ratio—embedded in rules of thumb of various trades and professions. On the one hand, we don't have to understand it to use it. Also now, mathematics gets built into products. Especially, very sophisticated mathematics can be incorporated in silicon, which will reliably perform computations that would boggle us but about which we, like the sixteenth-century coopers of Linz or the English nobility before the nineteenth century, need not bother ourselves.

On the other hand, some of us have to know this mathematics, or the rest of us cannot benefit from it. And we aren't sure beforehand who those some of us are. This huge gap, between the mathematics that most of us will obviously have to cope with on a daily basis and the mathematics that some of us have to know so that the rest of us can enjoy it in ignorance, is a major cause of the conundrum of prescribing mathematics for education.

So how much mathematics, and of what type, should every person learn today? Our democratic ideology is an important factor in these calculations. It demands that all students be educated to maximize their personal capacities. This ideal means that every student should be given a shot at qualifying for a job that really does require mathematics (since these are obviously the desirable jobs!). Advisors do not tell students that they should not take mathematics—on the contrary, as Secretary Riley properly does, they tell them that they should. For most students, any reason given, other than Secreatry Riley's, will be false (even, alas, the beauty reason, since beauty is in the eye of the beholder—if mathematical beauty were abundantely evident to the average person, Professor Dudley would have to use his sword to fend off students rather than to make shishkebab of anonymous NRC authors). It follows that many if not most students in the more advanced mathematics courses should not be there. They will not find it useful in further life, or beautiful, or otherwise worthwhile. They will decide not to take the next one. I think this situation is essentially unavoidable. Better teaching can ameliorate it but not eliminate it. The attrition rate throughout high school and college is widely quoted as 50 percent per year. If we double the effectiveness of our teaching system, a tremendous accomplishment of which we could justly be extremely proud, we would delay losses by only one year. Attrition in mathematics courses will continue to be a challenge to the profession. We should strive to improve mathematics instruction and mathematics curriculum. However, mathematics education will continue to serve as a sorting process.

Clearly, some waste occurs in this system. But we are talking about reproduction here—the reproduction of the mathematical expertise vital to the functioning of modern society—and reproduction is so important that vast resources are lavished on it. Think of how many maple seeds a maple tree produces each year to create a few viable offspring over its lifetime. How wasteful is our system of mathematics education? Consider calculus. Suppose that 1 percent of us will actually use calculus in our jobs (Which may be a wild overestimate, but it's a lot harder to debunk than 75%!). That 1 percent is actually a lot of people—over 30 000 a year. So if about 600 000 people per year are taking calculus, as many as 5 percent of those studying calculus may actually use it. (Probably most of these people are learning it in an AP high school course, which underlines the importance of getting this course right and of having good teachers at this level.) Perhaps two to three times as many will understand and enjoy it, and remember some of it. Even larger numbers will take away a few nice examples or lessons, or find it somewhat interesting at the time, or at least like their teacher. Also, a lot of these people are premedical students. Given the importance of technical expertise to society as a whole, and the doctrines of liberal education and of letting people discover their own limitations, I find these numbers broadly acceptable.

With all this said, I think that Professor Dudley is right. We should try to sell mathematics on the basis of its beauty, its power, its rigor. However, for me, the usefulness of mathematics, even in fairly mundane situations, is a part of its beauty. And since I know that students may not always on the first crack find the inspiring teacher who communicates the wonder of mathematics, I am glad that Secretary Riley is telling them how necessary it is as a credential.

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Reactions—Mercedes McGowen

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remedial mathematics courses, that is, arithmetic, algebra, and geometry (MAA 1997). Dropout rates as high as 50 percent in the traditional remedial courses have been cited (Hillel at al. 1992). According to the National Center for Education Statistics (1997), only 27 percent of students who enrolled in college completed four years despite the fact that 68 percent of incoming freshman at four-year

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Reactions—Mercedes McGowen

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colleges and universities had taken four years of mathematics in high school (NCES 1997). These students paid college tuition for courses that do not count for credit toward graduation at most colleges and universities.

What do we tell these students that makes sense to them about mathematics and why they should take mathematics courses? Dudley's conclusion that "mathematics is more important than jobs" and that "it transcends them" reflects the beliefs and experiences of a successful practicing mathematician. However, his experiences are not those of a majority of students studying mathematics today. The students enrolled in undergraduate remedial mathematics courses are commonly left with feelings of failure and a belief that mathematics is irrelevant, feelings very different from those described by Dudley. For these students, mathematics inspires fear not awe, discouragement not jubilation, a sense of hopelessness not amazement. Why do so many students who attempt rigorous mathematics courses not succeed? Even many of those who complete three or four years of "rigorous" high school mathematics are unsuccessful in subsequent college-level mathematics courses.

We need to have a clearer understanding of the differences and needs of the individual students in our classes, and we must take these differences into account in our curricular design and instructional practices. We have not yet figured out how to deal with those differences in ways that are appropriate to achieve the goal of mathematical power for all our students. The beliefs about what constitutes mathematics, what skills should be taught, when they should be taught, and to whom vary from individual to individual and community to community. The recent United States report on the Third International Mathematics and Science Study curriculum analysis (Beaton et al. 1996) cites these conflicting beliefs and practices, describing the current United States mathematics curriculum as unfocused, "a splintered vision," which is reflected in our mathematics curricular intentions, textbooks, and teacher practices. In comparison to other countries, the U.S. "adds many topics to its mathematics and science curriculum at early grades and tends to keep them in the curriculum longer than other countries do. The result is a curriculum that superficially covers the same topics year after year—a breadth rather than a depth approach."

Terms whose meanings were once commonly understood by those engaged in the practices of mathematics now have different meanings and serve as flashpoints for increasingly vehement discourse. Dialogue based on a common language and definitions has become extremely difficult, as Humpty Dumpty pointed out to Alice in *Through the Looking Glass:* "You see, it's like a portmanteau—there are two meanings packed up into one word."

In the absence of mutually agreed-on definitions and accepted meanings, the debate continues among those who favor a "return to basics" and those who are attempting to implement reforms in the teaching and learning of school mathematics, with increasingly high costs. Our vision has become not only fragmented but clouded by emotion. Witness the ongoing saga in California, where efforts to establish a set of statewide mathematics standards have generated contentious debate and vehemence on both sides.

Competing visions of what mathematics students should learn have polarized mathematics practitioners and educators, students, their parents, and the community at large. Robert Davis, in an electronic mail communication (1996), described the position in which we trap students: "There is at present a tug of war going on in education between a 'drill and practice and back to basics' orientation that focuses primarily on memorizing mathematics as meaningless rote algorithms vs. an approach based upon understanding and making creative use of mathematics." Does the current splintered vision of mathematics really serve the best interests of mathematicians, teachers, students, and the public? What do we really mean by "Algebra for All," and what mathematics should we be teaching? In our efforts to make mathematics accessible and attractive to a large number of students, are we, as Cuoco (1995) worries, "changing the very definition of mathematics?" Will it be a fundamentally different discipline in the future? Should it be a different discipline for some, for all, for none?

I agree that we should not tell students lies about why they should study mathematics. I also agree with the authors of "Mathematics Equals Opportunity" that students should study more mathematics earlier so as not to close off their options. However, I am no longer certain what it is that we should tell students. Even those who are successful in their mathematical endeavors in school often fail to recognize how what they learned in school is used in their work environment. The extent to which problem-solving skills and the use of symbols to mathematize situations are recognized in the workplace frequently goes unnoticed by employers as well. The assumption that algebra is the key to well-paying jobs and a competitive workforce requires the efforts of mathematically knowledgeable observers to support this assertion with data and to question the beliefs about what should be taught and how it should be taught in schools.

I disagree with Dudley's conclusion that mathematics is sufficient, not necessary. The failure to take rigorous mathematics courses has significant economic consequences in terms of future earnings, productivity, and stable employment. The United States Bureau of Labor Statistics (1997) predicts that, in the years between 1994 and 2005, occupations that require a bachelor's degree or above will average 23 percent growth, almost double the 12 percent growth projected for occupations that require less education and training, and that jobs requiring the most education and training will grow faster than jobs with lower education and training requirements. Typically, state colleges and universities recommend, and often require, that students take at least three years of mathematics in high school for entrance. Graduation requirements often include several more rigorous courses in mathematics or

Reactions—Mercedes McGowen

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science, and employers generally require applicants to pass standardized mathematics and reading tests.

The question is not "Is mathematics necessary?" but "What mathematics do we want students to learn?" and "How do we stop building Alban houses with windows shut down so close some students' spirits cannot see?" (Dickinson 1950). To answer those questions, we must find ways to reconcile the different understandings of mathematics held by parents ("What I learned in school"); by students ("A hoop to jump through" requirement for high school graduation, entry into college, college graduation); by teachers ("What's in the textbook"); and by such mathematicians as Dudley, for whom mathematics is the subject that is "the human race's supreme intellectual achievement"—that increases the ability to reason, inspiring awe, jubilation, and a sense of power and amazement."

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Reactions—Penny Noyce

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At the same time, "Mathematics Equals Opportunity" argues for the prime importance of algebra, and urges us to introduce algebra and geometry into the middle school or even earlier. What are we to make of the notion that first-year algebra is a "gateway" course? Is it just that the kinds of students who are likely to take first-year algebra early—higher-income students with involved parents—are the same group who are likely to attend college? Is it that success in algebra selects students capable of abstract reasoning? Or does algebra actually increase the ability to think, as Dudley asserts for all of mathematics?

I suspect that mathematics study is valuable because it accustoms us to a rigorous, quantitative approach to problems. But whatever the reason, algebra success is associated with college attendance; and major experimental interventions, chiefly the Equity 2000 project, are under way to determine whether extending algebra and geometry to poor and minority students does in fact increase their rate of college attendance. Still, we are confused about how best to move algebra and geometry down the grade levels. The temptation is to transfer the traditional high school course without adjustment, but such a strategy may crowd out the possibility of allowing middle school students to explore rich content in discrete mathematics, probability, or number theory. We may succeed only in creating more students who do mathematics but dislike it.

One barrier to progress in the mathematics curriculum is the conflict between what we want to do and how we count progress. For example, the most effective strategy for teaching advanced mathematics earlier may be to weave concepts of algebra and geometry through several years of the elementary and middle school curriculum, but what we count is the number of students taking a course called algebra in eighth grade.

Undoubtedly, much of the current upper elementary and middle school mathematics curriculum is repetitive and unchallenging. Students can do more. Diverse and challenging curriculum materials exist. Graphing calculators allow students, through experimentation, to reach an intuitive understanding of how functions behave with much less tedium and more delight than was true for those of us who once graphed everything by hand.

But if teachers are to use these new materials and tools successfully, they need to make significant changes in their teaching. Nor is change at one or two grade levels enough. High school, middle school, and even elementary teachers need to work together to plan, prepare, and evaluate a progression of courses, including integrated courses, that introduce not only algebra and geometry but probability and statistics, discrete mathematics, and number theory. Successful acceleration of all students, not just those most precocious at abstract thinking, will require long-term investment in professional development, examination of student work, and continual adjustment of teaching strategies to make sure that students actually understand. It won't happen overnight.

Reactions—John C. Souders Jr.

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most people to survive in today's high-technology world and the learning of advanced mathematics topics is unnecessary. For a while, I thought Dudley was basing this premise strictly on the operational mathematics required for most jobs and was ignoring the thought processes that mathematics acts as a catalyst to build. Then in the section "As a Way of Thought," Dudley makes a strong case that mathematics plays an important role in developing reasoning skills. However, since this assertion can be proved only anecdotally, he never definitively uses it as a counterbalance to his "math is not necessary" argument.

I would like to provide the counterbalance. The residual effect of studying mathematics is not a mastery of how to manipulate variables or perform operations on them. For most people, mathematics provides a platform for developing problem-solving skills that are rooted in logic and based on available facts. As people study more rigorous mathematics, their mastery of these skills continues to grow and their ability to discern cause-and-effect relationships sharpens. As businesses continue to flatten their organizational structures, employees will assume positions of greater responsibility and will be expected to solve problems more quickly and accurately. Therefore, any process that enhances problem-solving is valuable, indeed necessary, because it empowers people to reach their fullest potential as citizens and employees. If Dudley can assert without proof that mathematics transcends jobs, I feel comfortable stating that mathematics complements jobs.

Next, let me address the viewpoint expressed by Secretary Riley. Many hold the opinion that mathematics in today's world is essential and has meaning for the vast majority of our citizenry. Therefore, in principle, it is not difficult to support the Executive Summary and its proclamation that *all* high school students should take a rigorous mathematics curriculum. Even though this proclamation is timely and appropriate, is it practical and implementable in today's educational environment? Although many barriers stand in the way of successful implementation, one deserves special attention. Most of the rigorous mathematics taught in our schools is presented in an abstract manner. As research (e.g., Perkins [1995]) has shown, many of our students are not adept at abstract thinking. This mismatch must be addressed if all students are to take a rigorous mathematics curriculum.

The good news is that such progressive organizations as NCTM and, more recently, the American Mathematical Association of Two-Year Colleges have published standards that provide a framework for presenting mathematics from a more concrete perspective. Both sets of standards advocate a contextual, applied, and hands-on approach to presenting mathematics. This stance does not mean that these standards abandon abstract presentations; rather they seek a balance between abstraction and the concrete insertion of mathematical relevance and real-world experiences. Such balance is supported by research (e.g., Caine and Caine [1991]; Kolb [1984]) on learning styles and will address the needs of a much broader cross section of students. When this balance is achieved at the classroom level through new teaching and learning strategies, the proclamation can meet its goal and be truly viable for *all* students.

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Is Mathematics Necessary?

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We should not tell our students lies. They will find us out, sooner or later.

Besides, it demeans mathematics to justify it by appeals to work, to getting and spending. Mathematics is above that—far, far above. Can you recall why you fell in love with mathematics? It was not, I think, because of its usefulness in controlling inventories. Was it not instead because of the delight, the feelings of power and satisfaction it gave; the theorems that inspired awe, or jubilation, or amazement; the wonder and glory of what I think is the human race's supreme intellectual achievement? Mathematics is more important than jobs. It transcends them, it does not need them.

Is mathematics necessary? No. But it is sufficient.

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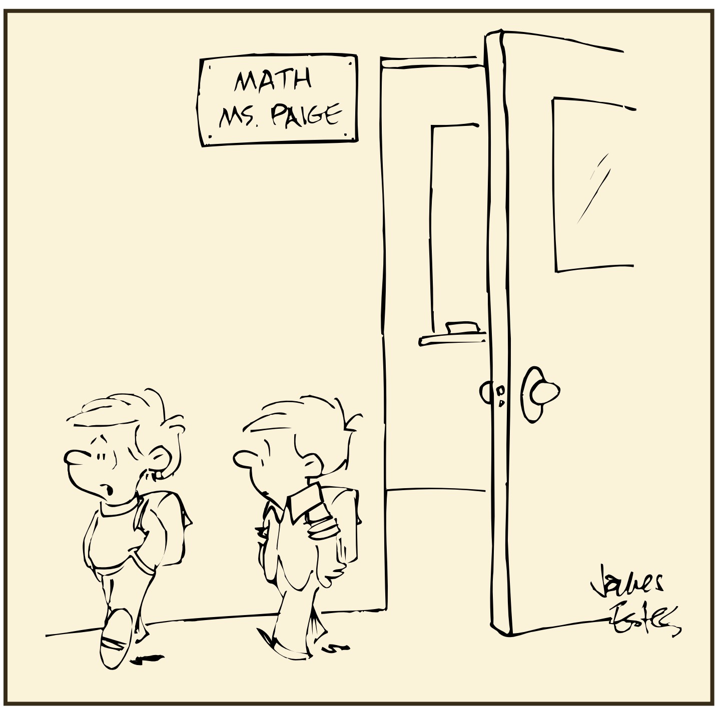
Is Long Division Obsolete?

In arithmetic, "long division" is an algorithm for finding the quotient of two numbers. The display of 4320 ÷ 75 shown below is typical and leads either to the quotient 57 with remainder 45 or to 57 45/75 or 57.6 if carried out to another decimal place. Long division is characteristically taught in fourth grade with whole numbers, and then in fifth or sixth grade with decimals. All books that teach long division start with one-digit divisors and then extend it to more digits.

$$\begin{array}{r} 57 \\ 75\overline{)4320} \\ \underline{375} \\ 570 \\ \underline{525} \\ 45 \end{array} \qquad \begin{array}{r} x+4 \\ x+1\overline{)x^2+5x+9} \\ \underline{x^2+x} \\ 4x+9 \\ \underline{4x+4} \\ 5 \end{array}$$



"Long division—something else I don't understand … I've just about resigned myself to leading a life of quiet desperation."

The NCTM's (Reston, Va.: NCTM 1989) *Curriculum and Evaluation Standards for School Mathematics* recommends that long division, and long division without remainders, be given decreased attention in K–4 mathematics (p. 21). Long division is not specifically mentioned in the 5–8 Standards, but they make the following statements: "Performing two-digit computations with whole numbers or decimals aids students in understanding connections between computation and numeration. Even though students can explore paper-and-pencil computations with numbers of any size and with various systems, they should not be expected to become proficient with paper-and-pencil computations with several digits. A curriculum that incorporated this standard would not include paper-and-pencil practice for proficiency with tedious computations, such as those with three-digit multipliers or divisors…" (p. 96).

A second and similar long-division algorithm is used with polynomials. The display of $(x^2 + 5x + 9) \div (x + 1)$ shown above at the left leads either to the quotient $x + 4$ with remainder 5 or to the quotient $x+4+\dfrac{5}{x+1}$.

The Standards do not mention this long division explicitly but say, "For college-intending students who can expect to use their algebraic skills more often [than other students], an appropriate level of proficiency remains a goal. Even for these students, however, available and projected technology forces a rethinking of the level of skill expectations" (p. 150). The technology that was projected in 1989 has appeared: two companies manufacture calculators that can symbolically divide polynomials and obtain the answers found by long division.

Is Long Division Obsolete?

What are your personal opinions about teaching these long-division algorithms, and why do you hold these opinions? Should the recommendations in the next version of the Standards with regard to long division be different from the recommendations given in 1989?

E-mail your comments to dialogues@nctm.org. You can also mail your response to "*Dialogues,* 1906 Association Drive, Reston, VA 20191-1593, or fax your response to NCTM at (703) 476-2970—Attention: *Dialogues.* Selected responses will appear in a future issue of *Dialogues.*

RESPONSES TO

“Is Long Division Obsolete?”

Susan Addington

Susan Addington is an associate professor of mathematics at California State University, San Bernardino. Her interests include preservice and in-service education for elementary and middle school teachers, writing mathematics for the World Wide Web, and presenting mathematics to the public.

Our Methodology Is Obsolete

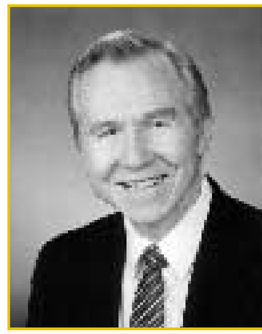
by Susan Addington

Long division is not obsolete, for pedagogical reasons: the process of long division, if carefully taught, can reinforce important elementary mathematical concepts and lead the way to advanced mathematics. Although calculators can compute quotients, they give the result as a truncated decimal, and, as a result, obscure concepts related to remainders and infinite sums. Specifically, long division has these advantages over “calculator division”:

• Long division gives a constructive (in the mathematical, not the educational, sense) way of obtaining a quotient of integers. A calculator is essentially a black box that gives the answer with no explanation. In particular, the long-division algorithm can reinforce the repeated-subtraction model of division; see the subsequent cookies problems.

• Long division naturally gives a whole-number remainder. A calculator doesn’t. (In actuality, some calculators do, but these remain hard to find—a busy parent shopping for school supplies at the supermarket won’t find one.)

See **Responses—Susan Addington,** page 17



Stephen S. Willoughby

Steve Willoughby has taught all grades from first through graduate school. He is interested in improving learning and teaching of mathematics at all levels. He has written more than 200 articles and books on mathematics and mathematics education. He is professor of mathematics at the Univeristy of Arizona and principal author of the K–6 series SRA Math: Explorations and Applications.

From the Pen of a K–16 Educator

by Stephen S. Willoughby

The Standards ought to be briefer and less prescriptive than they are, limited to describing the important things people should be able to do and the attitudes they should acquire toward mathematics. Among many other things, people should be proficient with basic facts and most multidigit algorithms; they should see mathematics as something they can figure out rather than memorize, and that is fun and useful rather than unpleasant and useless. They should also understand division. Whether they become proficient at long division is probably of very little consequence as long as they don’t spend too much time learning it.

Today, long division is analogous to square roots when I was teaching eighth grade in Massachusetts. The curriculum called for teaching square roots. I asked how the students knew that the square root of 25 was 5. They explained that 5 times 5 is 25. I asked them to find the square root of 30. They knew that 6 was too large so tried 5.5. That number was a bit too large. They tried 5.4. Too small. They

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Responses—Susan Addington

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Not only do remainders come up naturally in simple applied problems, but remainder arithmetic is also of central importance in such high-powered applications as cryptography and coding theory.

> ***Problem:*** How many 44-passenger school buses will 450 students need?

> ***Problem:*** Figure out a method to find the whole-number quotient and remainder using only a standard calculator. Try it on 15 263 794 divided by 3 572.

• Long division gives the first natural context in which you should regularly disbelieve your calculator. Different calculators will give the quotient of 158 by 9 as 17.555 555 56, 17.555 555, and 17.56. Which is right? Why does the apparent pattern of 5s in the first answer change to a 6? Why is the third answer so short? Students need to be aware that calculators round off, and that different calculators use different algorithms for the same calculation.

• Long division gives the first formal exposure to the concept of infinity—the algorithm generates infinite repeating-decimal expansions. This idea should be used as a prologue to calculus, since it is a convincing demonstration that an infinite set of numbers can have a finite sum.

Pedagogy

Long division has traditionally been the most disliked part of elementary arithmetic. Its many steps require careful attention to the rules of the procedure, neatness in aligning the digits, and sophisticated mental arithmetic. I suspect that the efficiency of this algorithm is an artifact of the days when students did their work on small slates—too many steps wouldn't fit. Here I offer some pedagogical suggestions for teachers of students who have lots of paper.

Since multiplication is repeated addition, division is repeated subtraction. You can convince second graders of this conclusion:

> ***Problem:*** You are filling bags with 7 cookies each. You have 35 cookies. How many bags can you fill?

```
 35
–  7  (fill one bag)
 28
–  7  (fill one bag)
 21
–  7  (fill one bag)
 14
–  7  (fill one bag)
  7
–  7  (fill one bag)
  0
```

Conclusion: 5 bags of 7, and no cookies left over

The method works just as well with large numbers, such as 15 263 794 divided by 3 572, although it might take a long time.

The "ladder" method, which I learned in the New Math era, is almost as transparent. Instead of subtracting the divisor, you subtract 10, or 100, or 1000 times the divisor, as large as possible. Keep track of what multiples of the divisor were subtracted in a column on the right.

> ***Example:*** 1619 divided by 7 has quotient 231 and remainder 2.

```
7)1619
  –700   100
  ----
   919
  –700   100
  ----
   219
   –70    10
  ----
   149
   –70    10
  ----
    79
   –70    10
  ----
     9
    –7     1
  ----   ---
     2   231
```

This method can be speeded up with exactly the kind of estimating that is required by the standard long-division algorithm. In fact, if you estimate optimally and leave out some of the bookkeeping, you have the standard algorithm.

```
7)1619                       231
 –1400   200              7)1619
 -----                     –14
   219                     ---
  –210    30                21
  ----                     –21
     9                     ---
    –7     1                 9
    --   ---                –7
     2   231                --
                             2
```

I see no need to "teach" long division using divisors of more than two digits. Such computations could be assigned (once!) as a problem of the week, or the ladder method could be used.

I conclude that it is not long division that is obsolete but the traditional method of teaching it mechanically. ●

Responses—Stephen S. Willoughby

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kept trying: 5.477 squared was 29.997529. That result was close enough.

The next day a student asked when we would learn the "real way" to find square roots. I asked what he meant. He started to show me the traditional algorithm but stopped after two or three steps. I asked why. "That's all my father remembers," he replied. I asked the class whether they would ever forget how to find a square root by guessing and multiplying. They agreed that they wouldn't, but they still wanted to learn the "real way" to do it. I agreed on the condition that they would help me explain why it works. Together we developed the standard geometric proof that the traditional algorithm produces answers as precise as needed. We agreed that the guess-and-check method produces equally precise answers but might take longer. We also learned the divide-and-average technique, sometimes known as *Newton's method,* and, at their insistence, discovered a cube-root algorithm based on analogous three-dimensional figures.

The student's father had forgotten the "real way" to find a square root because he didn't use it and probably never understood why it works. The students from my class probably no longer remember the traditional square root algorithm and also don't use it. But I hope that from that experience, and many others like it, they learned that mathematics is something to understand and think about rather than something to be memorized and regurgitated. I also hope that from many experiences in that class, they learned that mathematics can be fun and useful.

It is not evil to teach students to divide either whole numbers or polynomials. Nor is it evil to fail to teach these algorithms. But more knowledge is generally better than less.

Dividing by a single-digit number is easily taught by sharing some amount of play money "fairly" among several children I have in my "bank" only money with denominations that are powers of ten when doing so. By keeping records of the process, children arrive at the standard algorithm. Converting to the short-division algorithm is easy. By moving decimal points so that the divisor's decimal point is in the place after the first digit (*not,* generally, the rightmost digit), students estimate quotients no matter how many digits are in the divisor. If greater precision is needed, they estimate and multiply.

When, and if, polynomial division is learned, it should be related to the algorithm with whole numbers so that students see that connections occur in mathematics and that they can figure things out.

All New Forum Makes Debut

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to what extent do these learning styles imply that we need to teach them differently?

- What additional training, if any, do today's teachers of mathematics need to deal with these developments?

These questions introduce complex issues that do not lend themselves to simple answers. They have led to sometimes heated debates.

The Board of Directors of NCTM believes that reasoned discussion of questions like these is valuable for the entire education community: teachers, parents, students, administrators, and other concerned citizens. Recognizing that no vehicle has been specifically designed for this purpose, the Board formed a task force of six people (named below) to oversee the creation of two prototypes of a new publication that could serve as a forum for the identification and reasoned discussion of important issues in mathematics education.

This publication is the first of those two prototypes. We want your feedback. Please either copy and use the form (see page 19) or e-mail your comments to dialogues@nctm.org. Your reply can also be faxed to NCTM (attention: *Dialogues*) to (703) 476-2970 or sent by regular mail to *Dialogues,* 1906 Association Drive, Reston, VA 20191-1593.

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